MATH 3110: COMPLETE ORDERED FIELD AXIOMS

ZACH NORWOOD

These are the things we're assuming about the real numbers **R**. In mathematical jargon, they amount to saying that **R** is a *complete ordered field* with **Q** as a *subfield*. It is not important that you know what those words mean in general.

First, we assume that we have binary operations $+$ and $\cdot$ on the real numbers. We also assume that for each real number x there is a real number $-x$ called its *opposite* or *additive inverse*. We assume that for each real number $x \neq 0$ there is a real number $\frac{1}{x}$, called its *reciprocal* or *multiplicative inverse*. And we assume that there is an ordering $\leq$ defined on the real numbers.[1]

Finally, we assume that all the rational numbers are real numbers, including 0 and 1.

Field axioms. The following are true for all real numbers x, y, and z:

$$x + y = y + x \qquad \text{(addition is commutative)}$$
$$x + (y + z) = (x + y) + z \qquad \text{(addition is associative)}$$
$$x + 0 = x \qquad \text{(0 is an additive identity)}$$
$$x + (-x) = 0 \qquad \text{(additive inverses)}$$

$$x \cdot y = y \cdot x \qquad \text{(multiplication is commutative)}$$
$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \qquad \text{(multiplication is associative)}$$
$$x \cdot 1 = x \qquad \text{(1 is a multiplicative identity)}$$
$$\text{if } x \neq 0, \text{ then } x \cdot \frac{1}{x} = 1 \qquad \text{(multiplicative inverses)}$$
$$x \cdot (y + z) = x \cdot y + x \cdot z \qquad \text{(multiplication distributes over addition)}$$

Order axioms. The following are true for all real numbers x, y, and z:

$$x \leq x \qquad (\leq \text{ is a reflexive relation})$$
$$\text{if } x \leq y \text{ and } y \leq x, \text{ then } x = y \qquad (\leq \text{ is a antisymmetric relation})$$
$$\text{if } x \leq y \text{ and } y \leq z, \text{ then } x \leq z \qquad (\leq \text{ is a transitive relation})$$
$$x \leq y \text{ or } y \leq x \qquad (\leq \text{ is a total ordering})$$

$$\text{if } x \leq y \text{ and } z > 0, \text{ then } z \cdot x \leq z \cdot y \qquad \text{(multiplication and the order)}$$
$$\text{if } x \leq y, \text{ then } x + z \leq y + z \qquad \text{(addition and the order)}$$

Completeness Axiom. Every nonempty set with an upper bound has a *least upper bound*, i.e., a supremum.

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[1]By convention, we define the strict ordering as follows: $x < y$ iff $x \leq y$ and $x \neq y$.