THE TREE REFLECTION PRINCIPLE AND RESHAPING

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Recall that the Tree Reflection Principle $\mathrm{TRP}(\aleph_1)$, defined in [1], is the following assertion:

> For all $X \subseteq \omega_1$ and all trees $T \subseteq \omega_1$ of height ω_1, either T has a cofinal branch or $\{\alpha < \omega_1 : T \restriction \alpha$ has no cofinal branch in $L[X \cap \alpha]\}$ is stationary.

Theorem 1. $\mathrm{TRP}(\aleph_1)$ is equiconsistent with the existence of a weakly compact cardinal. In fact,

(a) if κ is weakly compact, then $\mathrm{TRP}(\aleph_1)$ holds in the extension by the Levy collapse to make $\kappa = \aleph_1$; and

(b) $\mathrm{TRP}(\aleph_1)$ implies that $\aleph_1$ is weakly compact in L.

Proof. (a) Let $V[G]$ be the extension by the Levy collapse. Kunen showed that $L(\mathbb{R})$-absoluteness for ccc posets holds in $V[G]$, and in [1] it is shown that $L(\mathbb{R})$-absoluteness for ccc posets implies $\mathrm{TRP}(\aleph_1)$. Alternatively, a routine Π^1_1-indescribability argument shows directly that $\mathrm{TRP}(\aleph_1)$ holds in $V[G]$.

For (b), suppose that $\kappa = \aleph_1$ is not weakly compact in L. By a theorem of Silver, this implies that there is a tree $T \in L$ of height κ with levels of size $< \kappa$ that has no uncountable branches in V. For this tree T and $X = \varnothing$, the principle $\mathrm{TRP}(\aleph_1)$ fails: since T is a thin tree, there is a club of $\alpha < \kappa$ for which $T \restriction \alpha$ includes the first α levels of T. For such α, any node of T on level α defines a branch through $T \restriction \alpha$, and this branch belongs to L since the entire tree belongs to L. ∎

Definition 2. A set $X \subseteq \omega_1$ is said to be *reshaped* if every $\alpha < \omega_1$ is countable in $L[X \cap \alpha]$.

If there is a club of α for which α is countable in $L[X \cap \alpha]$, then X can be modified to be reshaped. (Mimic the argument at the end of the proof of Theorem 7.)

Reshaped sets are typically used in conjunction with almost-disjoint coding to "code down to a real."

Definition 3. Let T be a tree of height α. We say that T is *pruned* if $T \restriction t$ has height α for every $t \in T$.

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The proof of the following lemma is routine.

Lemma 4.

(a) If T is a pruned tree, then a club of its subtrees are pruned.
(b) If T is a pruned tree of height α and α has countable cofinality, then T has a cofinal branch.

Theorem 5.

(a) If there is a reshaped subset of ω_1, then for every pruned tree T on ω_1 there is a set $X \subseteq \omega_1$ such that $\langle T, X\rangle$ witnesses the failure of $\mathrm{TRP}(\aleph_1)$.
(b) If there is a special tree T on ω_1 witnessing the failure of $\mathrm{TRP}(\aleph_1)$, then there is a reshaped subset of ω_1.

NB. We still do not assume that our trees are *thin*; that is, they could have uncountable levels.

Proof. (a) We use Lemma 4. Suppose that $X \subseteq \omega_1$ is reshaped, and let T be a pruned tree of height ω_1. We can assume that $T \restriction \alpha \in L[X \cap \alpha]$ for a club C of α, by (if necessary) using a definable pairing function $\langle\cdot,\cdot\rangle : \omega_1^2 \to \omega_1$ to make X code more information. We can also assume that $T \restriction \alpha$ is pruned for all $\alpha \in C$. Let $\alpha \in C$. The tree $T \restriction \alpha \in L[X \cap \alpha]$ is pruned and its height has countable cofinality in $L[X \cap \alpha]$, so $T \restriction \alpha$ has a cofinal branch in $L[X \cap \alpha]$. Thus $\langle T, X\rangle$ witnesses the failure of $\mathrm{TRP}(\aleph_1)$.

For (b), suppose that T is a special tree on ω_1, witnessed by a specializing function $f\colon T \to \omega$. Suppose further that X is a subset of ω_1 and $C \subseteq \omega_1$ is a club such that $\alpha \in C$ implies that $T \restriction \alpha$ has a cofinal branch in $L[X \cap \alpha]$. By replacing X with a set that codes more information and by shrinking C to a smaller club if necessary, we can assume that for all $\alpha \in C$,

- $T \restriction \alpha, f \restriction \alpha \in L[X \cap \alpha]$,
- $T \restriction \alpha$ has height α, and
- $L[X \cap \alpha] \vDash \alpha \leq \aleph_1$.

For all $\alpha \in C$, the tree $T \restriction \alpha$ is special and has a cofinal branch in $L[X \cap \alpha]$, so its height α must have countable cofinality in $L[X \cap \alpha]$. That is, α is countable in $L[X \cap \alpha]$. As mentioned above, a set that is "reshaped on a club" can easily be improved to a reshaped set, so we are done. ∎

Corollary 6. The nonexistence of reshaped subsets of ω_1 is equivalent to $\mathrm{TRP}(\aleph_1)$ for special, pruned trees.

Theorem 7. The nonexistence of reshaped subsets of ω_1 is equiconsistent with the existence of a Mahlo cardinal. In fact,

(a) if κ is Mahlo, then in the extension by the Levy collapse to make $\kappa = \aleph_1$ there is no reshaped subset of ω_1; and
(b) if there is no reshaped subset of ω_1, then $\aleph_1$ is Mahlo in L.

Proof. Suppose that κ is Mahlo and that $G \subseteq \mathrm{Coll}(\omega, <\kappa)$ is generic over V. Let $\dot{X}$ be a name for a set $X \subseteq \kappa$. There is (in V) a club of $\alpha < \kappa$ for which $\dot{X} \upharpoonright \alpha$ is a $\mathrm{Coll}(\omega, <\alpha)$-name; since κ is Mahlo in V, we can find such an α that is inaccessible, and thus $\mathrm{Coll}(\omega, <\alpha)$ has the α-cc. So α is a cardinal in the extension $V[G \cap \mathrm{Coll}(\omega, <\alpha)]$, and it is certainly also a cardinal in $L[X \cap \alpha]$, since $X \cap \alpha = (\dot{X} \upharpoonright \alpha)[G \cap \mathrm{Coll}(\omega, <\alpha)]$. So X is not reshaped.

Suppose that $\aleph_1$ is not Mahlo in L, so that there is a club $C \subseteq \omega_1$ of ordinals α for which $\mathrm{cf}^L(\alpha) < \alpha$. Build a set $X \subseteq \omega_1$ such that for every $\alpha \in C \cup \{0\}$ the segment $X \cap [\alpha, \alpha + \omega)$ codes a wellordering of the integers in ordertype α^{+C}, the next member of C. Now we prove by induction on $\alpha \in C$ that α is countable in $L[X \cap \alpha]$. The construction of X takes care of the successor case: $\alpha \in C \setminus \mathrm{Lim}(C)$. Suppose that α is a limit point of C. The inductive hypothesis ensures that every $\beta < \alpha$ is countable in $L[X \cap \alpha]$. But α is not a regular cardinal of L, so α must also be countable in $L[X \cap \alpha]$. ∎

Corollary 8. $\mathrm{TRP}(\aleph_1)$ for special, pruned trees is equiconsistent with the existence of a Mahlo cardinal.

References

[1] Itay Neeman and Zach Norwood. Coding along trees and generic absoluteness. *To appear*.