PROBLEM SETS 5–7: COMMENTS & SOLUTIONS

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Problem (PS 5 #3)**.** Consider the unit sphere S^2 in $\mathbb{R}^3$, meaning the set $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$ with the subspace topology. Let $\sim$ be an equivalence relation on S^2 given by $(x_1,y_1,z_1) \sim (x_2,y_2,z_2)$ if and only if $(x_1,y_1,z_1) = (-x_2,-y_2,-z_2)$. Show that the space $S^2/{\sim}$ is a manifold.

Solution. Let $\pi\colon S^2 \to S^2/{\sim}$ be the quotient map. We will to show that $S^2/{\sim}$ is Hausdorff and that every point in $S^2/{\sim}$ has an open neighborhood homeomorphic to R^2.

Consider distinct points $[x]$, $[y]$ in $S^2/{\sim}$. This means that x and y aren't antipodal, so the four points $x, -x, y, -y \in S^2$ are distinct.

In general: suppose that we have finitely many points $x_1, \dots, x_n$ in a subspace of $\mathbb{R}^n$. By taking $\epsilon < \inf\{\|x_i - x_j\| : i \neq j\}$, we get an $\epsilon > 0$ such that the open balls $B_\epsilon(x_i)$ are all disjoint.

So there is $\epsilon > 0$ such that (in particular) $U := B_\epsilon(x) \cup B_\epsilon(-x)$ and $V := B_\epsilon(y) \cup B_\epsilon(-y)$ are disjoint subsets of S^2. Put

$$\overline{U} = \big\{[u] \in S^2/{\sim} : \|u - x\| < \epsilon \text{ or } \|u-(-x)\| < \epsilon\big\} = \pi(U)$$

and

$$\overline{V} = \big\{[v] \in S^2/{\sim} : \|v - y\| < \epsilon \text{ or } \|v-(-y)\| < \epsilon\big\} = \pi(V),$$

and notice that $U = \pi^{-1}(\overline{U})$ and $V = \pi^{-1}(\overline{V})$. Since U and V are open in S^2, it follows (from the definition of the topology on $S^2/{\sim}$) that $\overline{U}$ and $\overline{V}$ are open in $S^2/{\sim}$. Suppose for a contradiction that $[w] \in \overline{U} \cap \overline{V}$. Then, replacing w by $-w$ if necessary, we conclude that $w \in U$ and $-w \in V$. But $\|-w-y\| = \|w-(-y)\|$, so $-w \in V$ if and only if $w \in V$. Since U and V are disjoint, such a w cannot exist. So $\overline{U}$ and $\overline{V}$ are disjoint open subsets of $S^2/{\sim}$, and $[x] \in \overline{U}$ and $[y] \in \overline{V}$. That is, $S^2/{\sim}$ is Hausdorff.

To prove that $S^2/{\sim}$ is 'locally Euclidean' we will split S^2 into three (definitely not disjoint) open sets, according to whether $x \neq 0$, $y \neq 0$, or $z \neq 0$. This will give a decomposition of $S^2/{\sim}$ into three open sets, and we will show that each of these sets is homeomorphic to $\mathbb{R}^2$.

Set $U = \{(x,y,z) \in S^2 : z \neq 0\}$. Since $(x,y,z) \in U$ iff $-(x,y,z) \in U$, we have $U = \pi^{-1}[\pi[U]]$ (U is the preimage of its image under U). Since U is an open subset of S^2, it follows that its image $\pi[U]$ is an open subset of $S^2/{\sim}$.

Date: 7 June 2013.

Define a map $h\colon U \to \mathbb{R}^2$ by $h(x,y,z) = (\frac{x}{z}, \frac{y}{z})$. (Geometrically, h does the following: for any point p on the sphere, there is a unique line through p and the origin, and, if the third coordinate of p is nonzero, that line intersects the plane $z = 1$ in a unique point. The map h sends p to the unique point on the plane $z = 1$ in this way.) This map h is continuous, by standard closure properties of the continuous functions $\mathbb{R}^m \to \mathbb{R}^n$. (It's continuous in each coordinate, since its coordinate functions are products of continuous functions.) The important observation is that $h(x,y,z) = h(-x,-y,-z)$, so the (well-defined) map

$$\overline{h}\colon \pi[U] \to \mathbb{R}^2, \quad \overline{h}([(x,y,z)]) = h(x,y,z) = (\tfrac{x}{z}, \tfrac{y}{z}),$$

is continuous. (See the theorem below.) The inverse of $\overline{h}$ is given by

$$\overline{h}^{-1}(x,y) = [\tfrac{1}{\sqrt{x^2+y^2+1}}(x,y,1)].$$

This inverse is also continuous, since it is the composite of the continuous functions

$$(x,y) \mapsto \tfrac{1}{\sqrt{x^2+y^2+1}}(x,y,1)$$

(which is continuous by standard closure properties of the continuous functions $\mathbb{R}^2 \to \mathbb{R}^3$) and the quotient map $\pi\colon S^2 \to S^2/{\sim}$. Therefore $\overline{h}$ is a homeomorphism $\pi[U] \to \mathbb{R}^2$.

There was nothing special about the choice of the third coordinate in that proof, so we could do the same thing for the set of $[(x,y,z)] \in S^2/{\sim}$ such that $y \neq 0$, or for the set of points such that $x \neq 0$. Since every point on S^2 has some nonzero coordinate (the origin is not on S^2!), every point in $S^2/{\sim}$ has an open neighborhood homeomorphic to $\mathbb{R}^2$. That is, $S^2/{\sim}$ is a manifold. ∎

We used the following important property of quotient spaces. If you didn't prove it in lecture, write down a proof yourself. It follows directly from the definition of the topology on the quotient space $X/{\sim}$.

Theorem. If $\sim$ is an equivalence relation on a space X, and $f\colon X \to Y$ is a continuous function constant on each equivalence class (meaning that for all $x, y \in X$, $x \sim y$ implies $f(x) = f(y)$), then there is a unique continuous map $\widetilde{f}\colon X/{\sim} \to Y$ such that $f = \widetilde{f} \circ \pi$. (Here $\pi\colon X \to X/{\sim}$ is the quotient map.)

Problem (PS 6 #6)**.** Consider the space $\mathbb{RP}^2 = S^2/{\sim}$ described in Problem Set 5, Problem 3. Define a path γ in $\mathbb{RP}^2$ from $[(1,0,0)]$ to itself by taking $\gamma(t) = [(\cos(\pi t), \sin(\pi t), 0)]$ for each $t \in [0,1]$. Show that the path $\gamma\gamma$ is homotopic to the constant path.

I'm disappointed that more people didn't give this problem a chance. I think it's a fun problem, and the main technical difficulty (parametrizing the ellipses to get the homotopy) is basically a 32A problem.

Solution. Let $\pi\colon S^2 \to S^2/{\sim}$ be the quotient map. The first thing to do is get a path $\widetilde{\gamma}$ in S^2 such that $\pi \circ \widetilde{\gamma} = \gamma\gamma$. (Caution: This is not possible for every path in $\mathbb{RP}^2$, but it happens to be for $\gamma\gamma$.) If you meditate on this for a minute, you will conclude that $\gamma\gamma$ is just the image of the map that traverses the equator of S^2 once in the counterclockwise direction. That is, we can define $\widetilde{\gamma}$ by

$$\widetilde{\gamma}(t) = (\cos(2\pi t), \sin(2\pi t), 0).$$

Let's prove carefully that $\pi \circ \widetilde{\gamma} = \gamma\gamma$. (These paths are actually **equal**, not just homotopic.) This boils down to the trig equalities

$$\cos(2\pi t - \pi) = \cos(2\pi t) = -\cos(2\pi t)$$

and

$$\sin(2\pi t - \pi) = -\sin(2\pi t).$$

From these it follows that

$$\begin{aligned}\gamma(2t-1) &= [(\cos(\pi(2t-1)), \sin(\pi(2t-1)), 0)]\\ &= [(-\cos(2\pi t), -\sin(2\pi t), 0)]\\ &= [(\cos(2\pi t), \sin(2\pi t), 0)]\\ &= \gamma(2t).\end{aligned}$$

With this in hand, we're ready to do the final calculation:

$$\begin{aligned}\pi\widetilde{\gamma}(t) &= \gamma(2t)\\ &= \begin{cases}\gamma(2t) & \text{if } t \in [0, \frac{1}{2}]\\ \gamma(2t-1) & \text{if } t \in [\frac{1}{2}, 1]\end{cases}\\ &= \gamma\gamma(t),\end{aligned}$$

by definition of the concatenation of two paths. The point is that $\pi\widetilde{\gamma}(t) = \gamma(2t)$ by definition of $\widetilde{\gamma}$, and $\gamma\gamma(t) = \gamma(2t)$ by the trig identities above.

Now we need to define a homotopy h from $\widetilde{\gamma}$ to the constant loop at $(1, 0, 0) \in S^2$, so then $\pi \circ h$ will be a homotopy from $\pi\widetilde{\gamma} = \gamma\gamma$ to the constant loop at $\pi(1, 0, 0) = [(1, 0, 0)]$. This technique can be used to show that any loop in S^2 is homotopic to the constant loop at its basepoint.

The idea is to pull the loop $\widetilde{\gamma}$, which loops around the equator, up over the sphere to the constant loop. Imagine stretching a rubber band around a soccer ball while holding one end of the rubber band. Then slowly bring the other end of the rubber band over the top of the ball until it meets the end you're holding in place. At time s of the homotopy, the loop should traverse an ellipse on the sphere whose shadow in the xy plane is a circle of radius $1-s$ centered at $(s,0)$. How do you parametrize a circle of radius $1-s$ and center $(s,0)$? Remember 32A! You want $(\frac{x-s}{1-s})^2 + (\frac{y}{1-s})^2 = 1$, so put $\cos(2\pi t) = \frac{x-s}{1-s}$ and $\sin(2\pi t) = \frac{y}{1-s}$ and solve for x and y. Then z should be $\sqrt{1-(x^2+y^2)}$, so we get

$$h(s,t) = (x, y, \sqrt{1-(x^2+y^2)}),$$

where

$$x = (1-s)\cos(2\pi t) + s \quad \text{and} \quad y = (1-s)\sin(2\pi t).$$

Then $h\colon [0,1]\times[0,1] \to S^2$ is continuous, as usual by standard closure properties of functions between subspaces of $\mathbb{R}^n$. We calculate to see that

$$\begin{aligned} h(0,t) &= (\cos(2\pi t), \sin(2\pi t), 0) = \widetilde{\gamma}(t), \\ h(1,t) &= (1,0,0), \\ h(s,0) &= (1,0,0), \\ h(s,1) &= (1,0,0) \end{aligned}$$

for all $s,t \in [0,1]$. The first two equations show that h is a homotopy from $h(0,-) = \widetilde{\gamma}(t)$ to the constant loop to $(1,0,0)$, and the last two equations show that h is a basepoint-fixing homotopy. We conclude that $\gamma\gamma = \pi \circ \widetilde{\gamma}$ is homotopic (by $\pi \circ h$) to the constant loop at $[(1,0,0)] = \pi(1,0,0)$. ∎

Problem (PS 7 #1)**.** Let X, U, and E be path-connected topological spaces, and suppose that $p_u\colon U \to X$ and $p_e\colon E \to X$ are both covering maps. Suppose that U is simply connected, meaning that for any $a \in U$ the group $\pi_1(U,a)$ only has one element. Suppose also that U is locally path-connected, meaning that for every point $x \in U$, every open nbhd V of x contains an open nbhd W of x (so $x \in W \subseteq V$) such that W is path-connected. Show that there is a covering map $p\colon U \to E$ with $p_e \circ p = p_u$.

In discussion section and in an email, I gave (more or less) the following outline for a solution to this problem.

(1) Pick special points $x_0 \in U$ and $e_0 \in p_e^{-1}(p_u(x_0))$. We show how to define $p(x)$ for $x \in U$. Since U is path-connected, there is a path γ in U such that $\gamma(0) = x_0$ and $\gamma(1) = x$. By the Path Lifting Lemma, there is a unique lift of the path $p_u \circ \gamma$ to a path in E with initial point e_0. That is, there is a unique path $\alpha\colon [0,1] \to E$ such that $\alpha(0) = e_0$ and $p_e \circ \alpha = p_u \circ \gamma$. Define $p(u) = \alpha(1)$.

Remark: p is already well-defined! We made a global choice ('choice', as in 'axiom of choice') of path γ (depending on x — maybe γ_x would've been more suggestive) from x_0 to x, and used that choice to define $p(x)$.

The uniqueness of the lift α guarantees that $p(x)$ is uniquely determined from x (and the choice of all of the paths γ).

(2) It is already easy to show that $p_e \circ p = p_u$: with the setup as in the definition of p, we have

$$p_e(p(x)) = p_e(\alpha(1)) = p_u(\gamma(1)) = p_u(x).$$

(3) Prove using the Homotopy Lifting Lemma that the definition of $p(x)$ does not depend on the choice of the path γ from x_0 to x.

(4) Prove that p is continuous. This is a bit tricky. To do it, prove the following claim:

Claim. Every point $x \in U$ has an open nbhd V such that $p_u(V)$ is contained in some open subset W of X that is evenly covered by p_e, and $p(V)$ is contained entirely in one connected component of $p_e^{-1}(W)$.

You will need (3) and the assumption that U is locally path-connected. To deduce from the claim that p is continuous, show that $p \restriction V$ is continuous, for V as in the claim. It is generally true that a function $f\colon X \to Y$ is continuous iff every point in X has an open nbhd O such that the restriction $f \restriction O$ is continuous.

(5) Prove that p is a covering map. That is, prove that every point $a \in E$ has an open neighborhood that is evenly covered by p. To do this, you should need to use that both p_e and p_u are covering maps.

Problems 2 & 3 on this problem set aren't too bad, but let's take a look at problem 4.

Problem (PS 7 #4)**.** Suppose that $p\colon U \to X$ is a covering map, X and U are both path-connected, and U is simply-connected and locally path-connected. Let b be a point in X, and $\widetilde{b}$ a point in U with $p(\widetilde{b}) = b$. Given a loop γ in X from b to b, show that there is a deck transformation $\phi_\gamma\colon U \to U$ with $\phi_\gamma(\widetilde{b}) = \widetilde{\gamma}(1)$, where $\widetilde{\gamma}$ is the unique lift of γ with $\widetilde{\gamma}(0) = \widetilde{b}$.

Notice that the proof of problem 1 actually gives the following stronger version:

Problem (Problem $1'$)**.** Setup is as in problem 1, but suppose we have chosen specific points $x_0 \in U$ and $e_0 \in E$ satisfying $p_u(x_0) = p_e(e_0)$. Then p can be chosen so that $p(x_0) = e_0$.

Prove the following lemma and combine it with problem $1'$ to get a solution to problem 4.

Lemma. Suppose that E is path-connected, $p\colon E \to X$ is a covering map, and $e_0, e_1 \in E$. Suppose also that $\phi, \psi\colon E \to E$ are continuous maps satisfying

$\phi(e_0) = e_1$, $\psi(e_1) = e_0$, and $p = p \circ \phi = p \circ \psi$. Then $\psi \circ \phi = \phi \circ \psi = \mathrm{id}_E$, so ϕ and ψ are deck transformations.

To prove the lemma, pick a path α in E from e_0 to a typical point $x \in E$ and show that $\psi \circ \phi \circ \alpha$ is also a lift of the path $p \circ \alpha$ with the same initial point. Then appeal to uniqueness of lifts. (The argument is symmetric in ϕ and ψ.)