INTEGRATION BY PARTS AND TRIG SUBSTITUTION

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1. Standard by-parts integrals

These are the integrals that will be automatic once you have mastered integration by parts. In a typical integral of this type, you have a power of x multiplied by some other function (often e^x, $\sin x$, or $\cos x$). Let u be the power of x and v' be the other function so that integrating by parts decreases the power of x.

Example 1. Compute $\int x \sin x \, dx$.

We use the substitution

$$\begin{aligned} u &= x \quad & v &= -\cos x \\ u' &= 1 \quad & v' &= \sin x. \end{aligned}$$

Then integrate by parts:

$$\int x \sin x \, dx = -x\cos x - \int (-\cos x) \, dx = -x \cos x + \sin x + C.$$

Other examples of integrals of this type:

- $\int x^2 e^x \, dx$
- $\int (2x)^2 \cos x \, dx$
- $\int x \sin(2x) \, dx$

Don't be frightened by the constants. They don't affect the method at all: you integrate $\int x^2 \cos x \, dx$ and $\int (3x/2)^2 \cos(3x) \, dx$ using the same method; the constants are just different.

2. Tricky by-parts integrals

What makes these integrals strange is that setting $v' = 1$ is often a good idea. Also, the integrand is often not a product, as you will see in these examples.

Example 2. Compute $\int \ln(x) \, dx$.

We use the substitution

$$\begin{aligned} u &= \ln(x) \quad & v &= x \\ u' &= \tfrac{1}{x} \quad & v' &= 1. \end{aligned}$$

Then integrate by parts:

$$\int \ln(x) \, dx = x \ln(x) - \int x \tfrac{1}{x} \, dx = x\ln(x) - \int 1 \, dx = x \ln(x) - x + C.$$

In that example, somehow the extra factor x you get by integrating $v' = 1$ cancels out with $u' = \frac{1}{x}$ nicely.

Example 3. Compute $\int \arcsin(x) \, dx$.

We use the substitution

$$\begin{aligned} u &= \arcsin x \quad & v &= x \\ u' &= \frac{1}{\sqrt{1-x^2}} \quad & v' &= 1. \end{aligned}$$

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Then integrate by parts:

$$\int \arcsin x \, dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} \, dx. \tag{1}$$

The integral on the right is a typical u-substitution integral. Set $u = 1 - x^2$ to get $du = -2xdx$ and

$$\int \frac{x}{\sqrt{1-x^2}} \, dx = -\frac{1}{2} \int \frac{du}{\sqrt{u}} = -\frac{1}{2} \int u^{-1/2} = -\sqrt{u} + C = -\sqrt{1-x^2} + C.$$

Plug this result back into equation (1) to get

$$\int \arcsin x \, dx = x \arcsin x - (-\sqrt{1-x^2}) + C = x \arcsin x + \sqrt{1-x^2} + C.$$

This didn't work out quite as nicely as Example 2 did, but the x we got by integrating $v' = 1$ served as (part of) the du in our substitution.

For another tricky by-parts integral, try $\int (\ln x)^2 \, dx$.

3. Sneaky by-parts integrals

The main example of this type of integral is the following:

Example 4. Compute $\int e^x \cos x \, dx$.

We use the substitution

$$\begin{aligned} u &= e^x & v &= \sin x \\ u' &= e^x & v' &= \cos x. \end{aligned}$$

Then integrate by parts:

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

Someone who's paying attention to what (s)he is doing at this point might say, 'Well, we haven't gotten anywhere, since $\int e^x \sin x \, dx$ is no easier than the integral we started with!'. That's a reasonable response, but let's charge ahead anyway. Use another substitution for the integral on the right:

$$\begin{aligned} u &= e^x & v &= -\cos x \\ u' &= e^x & v' &= \sin x. \end{aligned}$$

Integrating by parts a second time gives

$$\int e^x \cos x \, dx = e^x \sin x - (-e^x \cos x + \int e^x \cos x \, dx) = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx.$$

Here's where the sneakiness comes in. The integral on the far right is now our original integral, so we can add it to both sides and divide by 2 to get a formula for the original integral!

$$2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C,$$

and dividing by 2 gives

$$\int e^x \cos x \, dx = \tfrac{1}{2}(e^x \sin x + e^x \cos x) + C.$$

This phenomenon is difficult to replicate (other than in obvious variants of the example, like $\int e^x \sin x \, dx$ or $\int e^{(2x)} \sin(3x) \, dx$). As a result, most problems that require this sneaky trick will look like $\int e^x \cos x \, dx$ or $\int e^x \sin x \, dx$ (possibly with extra constants, of course). (One important exception is $\int \sec^3 x \, dx$, though; see below.)

4. Trig integrals

Before we do some nastier by-parts integrals, we need to learn some trig integrals. First, an example that you've known how to do for a while:

Example 5. Compute $\int \sin^3 x \cos x \, dx$.

We notice that the substitution $u = \sin x$, $du = \cos x \, dx$ simplifies the integral considerably:

$$\int \sin^3 x \cos x \, dx = \int u^3 \, du = \frac{u^4}{4} + C = \frac{\sin^4 x}{4} + C.$$

Example 6. Compute $\int \sec x \, dx$.

5. Extra tricky (and sneaky) by-parts integrals

Example 7. Compute $\int \sec^3 x \, dx$.

6. Exercises

When you've mastered the examples in the previous few sections, try these:

(1) $\displaystyle\int \sin \sqrt{x} \, dx$.

(2) $\displaystyle\int x \ln x \, dx$.

(3) $\displaystyle\int \frac{1}{t - \sqrt{1 - t^2}} \, dt$.

(4) $\displaystyle\int \arcsin \sqrt{x} \, dx$.

(5) $\displaystyle\int \frac{1}{x^4 + 4} \, dx$.

(6) $\displaystyle\int \sin(\ln x) \, dx$.

(7) $\displaystyle\int \cos x \ln(\sin x) \, dx$.

(8) $\displaystyle\int \sin x \ln(\sin x) \, dx$.