PRACTICE FINAL SOLUTIONS (SPRING 2012)

**Problem 1.**

(a) ...every disk centered at $(a, b, c)$ contains points in $D$ and points not in $D$.

(b) It is the following limit, if it exists:

$$\lim_{t \to 0} \frac{f(a - \frac{1}{2}t, b - \frac{1}{2}t, x + \frac{1}{2}t, d + \frac{1}{2}t) = f(a, b, c, d)}{t}.$$

(c) ... $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

(d)

$$\lim_{(x,y)\to(a,b)} \frac{e(x, y)}{\sqrt{(x - a)^2 + (y - b)^2}} = 0,$$

where $e(x, y) = f(x, y) - L(x, y)$.

(e) (i) illegal;
(ii) scalar;
(iii) vector;
(iv) illegal.

**Problem 2.**

(a)

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{3\cos t}{-\sin t} = -3\frac{\cos t}{\sin t} = -3\cot t.$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{x'(t)} = \frac{3\csc t}{-\sin t} = -\frac{3}{(\sin t)^3}. \checkmark$$

(b) First notice that the two lines given intersect in the point $(-1, 2)$, which is therefore the center of the hyperbola. There are many correct answers. For any $a, b$ such that $\frac{b}{a} = 2$,

$$\left(\frac{x + 1}{a}\right)^2 - \left(\frac{y - 2}{b}\right)^2 = 1$$

is correct. (Take, for instance, $a = 7$ and $b = 14$.)

(c) The key insight: four points are coplanar if and only if the parallelepiped they span has volume 0. Recall that you can compute the volume of this parallelepiped using the scalar triple product, as follows. We need three vectors, each starting at one of the given points and ending at another:

$$\begin{aligned}\vec{v} &= \overrightarrow{AB} = \langle 1, 1, -1\rangle \\ \vec{w} &= \overrightarrow{AC} = \langle 2, 3, -2\rangle \\ \vec{u} &= \overrightarrow{AD} = \langle 4, 5, -4\rangle .\end{aligned}$$

Now we compute the scalar triple product $|\vec{u}\cdot(\vec{v}\times\vec{w})|$ by first computing $\vec{v}\times\vec{w}$:

$$\vec{v}\times\vec{w} = \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 2 & 3 & -2 \end{vmatrix} = \langle 1, 0, 1\rangle .$$

Now we have

$$|\vec{u}\cdot(\vec{v}\times\vec{w})| = |\vec{u}\cdot\langle 1,0,1\rangle| = 0,$$

so the volume of the parallelepiped is 0. That is, the four points are coplanar. ✓

(d) First we simplify the limit:

$$\lim_{(x,y)\to(1,2)} \frac{xy-2x-y+2}{x^2+y^2-2x-4y+5} = \lim_{(x,y)\to(1,2)} \frac{(x-1)(y-2)}{(x-1)^2+(y-2)^2}.$$

Now compute the limit along lines $y = m(x-1)+2$ of slope $m$ through the point $(1,2)$:

$$\lim_{\substack{(x,y)\to(1,2)\\ \text{along } y=m(x-1)+2}} \frac{(x-1)(y-2)}{(x-1)^2+(y-2)^2} = \lim_{x\to 0} \frac{(x-1)(m(x-1))}{(x-1)^2+m^2(x-1)^2} = \frac{m}{1+m^2}.$$

Certainly $\frac{m}{1+m^2}$ depends on which $m$ we choose; for instance, we get 0 with $m=0$ and $\frac{1}{2}$ with $m=1$. So the limit is different along different lines through $(1,2)$, and therefore the limit doesn't exist. ✓

**Problem 3.**

(a) We need a normal vector for the plane. Since the plane should contain the line $L_1$, its normal vector should be orthogonal to any direction vector for $L_1$. Moreover, the vector from $A$ to any point on $L_1$ should also be orthogonal to the normal vector for the plane. By plugging in $t=0$, we see that the point $(-1,0,0)$ lies on the line $L_1$. So we can get a normal vector for the plane by taking the cross product of $\overrightarrow{AB}$ and $\langle -1,2,-1\rangle$, the direction vector for the line:

$$\vec{n} = \begin{vmatrix} i & j & k \\ 2 & 2 & 0 \\ 1 & 2 & -1 \end{vmatrix} = \langle -2, 2, 2\rangle .$$

So our plane's equation is $-2x+2y+2z=d$; to find $d$, we just plug in the coordinates of one of the points that should be on the plane. For example, use $(1,2,0)$ to get $d = -2(1) + 2(2) + 2(0) = 2$. So the plane's equation is $-2x + 2y + 2z = 2$, i.e., $-x + y + z = 1$. ✓

(b) Following the hint, we find a point $B$ on the line $L_1$ such that $\overrightarrow{AB}$ is perpendicular to $L_1$. To do this, first pick any point $B'$ on the line, and project the vector $\overrightarrow{AB'}$ onto a direction vector for the line. (Draw a picture.) An easy choice of point on the line is $(-1,0,0)$ from part (a), so $\overrightarrow{AB} = \langle -2,-2,0 \rangle$. The projection of $\langle -2,-2,0 \rangle$ onto $\langle 1,2,-1 \rangle$ is given by

$$\frac{\langle 1,2,-1 \rangle \cdot \langle -2,-2,0 \rangle}{\|\langle 1,2,-1 \rangle\|^2} \langle 1,2,-1 \rangle = \langle -1,-2,1 \rangle .$$

The direction vector $\vec{u}$ of our line $L_2$ should be perpendicular to $L_1$, so we should have $\vec{n} = \overrightarrow{AB'} - \langle -1,-2,1 \rangle$. That is, $\vec{n} = \langle -2,-2,0 \rangle - \langle -1,-2,1 \rangle = \langle -1,0,-1 \rangle$. We have a point and a direction vector, so we can write down the parametrization of the line:

$$\langle 1,2,0 \rangle + t \langle -1,0,-1 \rangle . \checkmark$$

**Problem 4.**

(a) The projection is given by

$$\mathrm{proj}_{\vec{b}}(\vec{a}) = \frac{\langle 1,2,3 \rangle \cdot \langle 1,1,1 \rangle}{\|\langle 1,1,1 \rangle\|^2} = \frac{1+3+2}{3} \langle 1,1,1 \rangle = \langle 2,2,2 \rangle .$$

(b) First, we notice that the intersection of the line and the plane is the point $(0,2,-2)$. To see this, substitute the formulas for $x$, $y$, and $z$ in the definition of the line for $x$, $y$, and $z$ in the equation for the plane, obtaining

$$(-1+t) + (-1+3t) + (-4+2t) = 0.$$

Solving this gives $t = 1$, and the point on the line corresponding to $t = 1$ is $(0,2,-2)$.

The vector $\langle 1,1,1 \rangle$ is a normal vector for the plane, so the projection $\vec{n}$ computed in part (a) is also a normal vector to the plane. Write $\vec{a}$ for the vector $\langle 1,3,2 \rangle$, a direction vector for the line $L_1$. If you draw in $\vec{a}$ and $\vec{n}$ on the picture, you will see that

$$\vec{a} - 2\vec{n} = \langle 1,3,2 \rangle - 2 \langle 2,2,2 \rangle = \langle -3,-1,-2 \rangle$$

is a direction vector for $L_2$. So $L_2$ is parametrized by

$$\langle 0,2,-2 \rangle + t \langle -3,-1,-2 \rangle . \checkmark$$

**Problem 5.**

(a) The angle between two planes is defined to be the angle between their normal vectors.

(b) The normal vector for our plane should be orthogonal both to the normal vector $\langle 2, -3, 1\rangle$ of the given plane and to $\langle -3, 1, 2\rangle$, the vector $\overrightarrow{(-1,1,-1)(2,2,1)}$. So we just take the cross product of these two vectors:

$$\langle 2, -3, 1\rangle \times \langle 3, 1, 2\rangle = \begin{pmatrix} i & j & k \\ 2 & -3 & 1 \\ 3 & 1 & 2 \end{pmatrix} = \langle -7, -1, 11\rangle .$$

So our plane has equation $-7x - y + 11z = d$. We find $d$ by plugging in some point we want to lie on the plane; for instance, plug in $(2, -2, 1)$ to get $d = -7(2) - (-2) + 11(1) = -5$. So the equation of the plane is $-7x - y + 11z = -5$. ✓

**Problem 6.**

(a) A level curve is the projection to the $xy$-plane of a horizontal trace.

(b) We want to fix $x$; say $x = a$. The equation becomes $z = 3a^2 - y^2 - 2ay$. This is a parabola (facing downward). If $y$ is fixed (say $y = b$), then the equation becomes $z = 3x^2 - 2xb - b^2$, which also defines a parabola (facing upward).

(c) The horizontal trace at $z = -5$ is defined by

$$-5 = f(x, y) = 3x^2 - y^2 - 2xy.$$

Recall that the gradient of $f(x, y)$ at $(1, 2)$ is normal to the level curve at $(1, 2)$, so we can compute the gradient and then use that to get a direction vector for the tangent line.

$$\nabla f = \langle 6x - 2y, -2y - 2x\rangle ; \quad \nabla f(1, 2) = \langle 2, -6\rangle .$$

So $\langle 1, 2, -5\rangle + t\,\langle 6, 2, 0\rangle$ parametrizes the line tangent to the horizontal trace through $(1, 2, -5)$. ✓

**Problem 7.**

(a) The derivative on each piece is continuous, so it suffices to show that the derivative functions on the pieces 'line up'.

The derivative on the first piece is 0, and the derivative on the cubic piece is $x^2$. These have the same value at 0, so the derivative is continuous at 0. The derivative of the cubic piece at 1 is 1, so we need to check that the derivative of the circular piece at $x = 1$ is also 1. The relevant portion of the circle is parametrized by $\mathbf{r}(t) = \left\langle 2\cos t, \frac{4}{3} + 2\sin t\right\rangle$ for $t \in [-\frac{\pi}{4}, 0]$. We have $\mathbf{r}'(t) = \langle -2\sin t, 2\cos t\rangle$, so $\mathbf{r}'(-\frac{\pi}{4}) = \langle 1, 1\rangle$. So the limit as $x$ approaches 1 of the derivative of the circular piece is also 1, which means that the derivative function is also continuous at 1. ✓

(b) First observe that $\mathbf{r}''(t) = \langle -1, 1\rangle$, so we can compute the curvature of the circular piece:

$$\kappa(-\tfrac{\pi}{4}) = \frac{\left\|\mathbf{r}'(-\frac{\pi}{4}) \times \mathbf{r}''(-\frac{\pi}{4})\right\|}{\left\|\mathbf{r}'(-\frac{\pi}{4})\right\|^3} = \frac{2}{(\sqrt{2})^3} = \frac{1}{\sqrt{2}}.$$

The curvature of the cubic piece is given by:

$$\kappa(x) = \frac{|2x|}{(1+(x^2)^2)^{3/2}} = \frac{|2x|}{(1+x^4)^{3/2}},$$

which is continuous for all $x$ (since the denominator is never 0) and has value $\frac{1}{\sqrt{2}}$ at $x = 1$. So the curvature function is continuous at $x = 1$. For the cubic piece, we also have $\kappa(0) = 0$, so the curvature function is also continuous at 0. Since the curvature function of each piece is continuous on the piece, the whole curvature function is continuous. ✓

**Problem 8.**

(a) First we compute $\mathbf{r}'$ and $\mathbf{r}''$:

$$\mathbf{r}'(t) = \langle -4\sin t, 3, 4\cos t\rangle$$
$$\mathbf{r}''(t) = \langle -4\cos t, 0, -4\sin t\rangle .$$

Computing $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ is just a matter of remembering the definitions and computing:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\langle -4\sin t, 3, 4\cos t\rangle}{\sqrt{(-4\sin t)^2 + 3^2 + (4\cos t)^2}} = \frac{1}{5}\langle -4\sin t, 3, 4\cos t\rangle .$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{\frac{1}{5}\langle -4\cos t, 0, -4\sin t\rangle}{4/5} = \langle -\cos t, 0, -\sin t\rangle .$$

$$\mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t) = \frac{1}{5}\begin{vmatrix} i & j & k \\ -4\sin t & 3 & 4\cos t \\ -\cos t & 0 & -\sin t\end{vmatrix} = \frac{1}{5}\langle -3\sin t, -4, -3\cos t\rangle .$$

(b) The normal plane has tangent vector $\mathbf{T}(t)$, and $\mathbf{r}(t)$ lies on it. So it has equation

$$(-\tfrac{4}{5}\sin t)(x - 4\cos t) + \tfrac{3}{5}(y - 3t) + \tfrac{4}{5}\cos t(z - 4\sin t) = 0.$$

If you want, clear denominators and rearrange to get

$$(-4\sin t)x + 3y + (4\cos t)z = 9t - 16\cos t + 16\sin t\cos t.$$

**Problem 9.** We have the function $f(x,y,z) = xy + yz$ and the constraints $g(x,y,z) = x^2 + y^2 - 4 = 0$ and $h(x,y,z) = yz - 4 = 0$. First compute gradients:

$$\nabla f = \langle y, x+z, y\rangle$$
$$\nabla g = \langle 2x, 2y, 0\rangle$$
$$\nabla h = \langle 0, z, y\rangle .$$

Now use the Lagrange method to get equations:

$$\langle y, x+z, y\rangle = \nabla f = \lambda\nabla g + \mu\nabla h = \lambda\langle 2x, 2y, 0\rangle + \mu\langle 0, z, y\rangle .$$

This gives the following three equations:

$$y = 2x\lambda \tag{1}$$
$$x + z = 2y\lambda + \mu z \tag{2}$$
$$y = y\mu. \tag{3}$$

From the third equation we deduce that either $y = 0$ or $\mu = 1$.

**Case 1:** $y = 0$. We rule out this case, since no point with $y$-coordinate 0 satisfies the constraint $h(x, y, z) = yz - 4 = 0$. ✓

**Case 2:** $\mu = 1$. Substituting 1 for $\mu$ in equation (2) and canceling the $z$'s gives $x = 2y\lambda$. Plug this in for $x$ in equation (1) to get $x = 4x\lambda^2$, which implies that either $x = 0$ or $\lambda^2 = \frac{1}{4}$ (so $\lambda = \pm\frac{1}{2}$).

**Case 2a:** $x = 0$. We have $y = 2x\lambda = 0$, but no point with $x$-coordinate 0 and $y$-coordinate 0 satisfies the constraint $g(x, y, z) = x^2 + y^2 - 4 = 0$. So this case yields no solutions.

**Case 2b:** $x \neq 0$ and $\lambda = \pm\frac{1}{2}$. In this case, $y = \pm x$, so we can solve the constraint equation $x^2 + y^2 = 4$ to get $2x^2 = 4$, i.e., $x = \pm\sqrt{2}$ and $y = \pm\sqrt{2}$. If $y = \pm\sqrt{2}$, then the other constraint gives $\pm\sqrt{2}z - 4 = 0$, i.e., $z = \pm 2\sqrt{2}$. We have four points to consider:

$$(\sqrt{2}, \sqrt{2}, 2\sqrt{2}),\ (-\sqrt{2}, -\sqrt{2}, -2\sqrt{2}),\ (-\sqrt{2}, \sqrt{2}, 2\sqrt{2}),\ \text{and } (\sqrt{2}, -\sqrt{2}, -2\sqrt{2}).$$

Check that

$$f(\sqrt{2}, \sqrt{2}, 2\sqrt{2}) = f(-\sqrt{2}, -\sqrt{2}, -2\sqrt{2}) = 6, \text{ and}$$
$$f(-\sqrt{2}, \sqrt{2}, 2\sqrt{2}) = f(\sqrt{2}, -\sqrt{2}, -2\sqrt{2}) = 2.$$

So the maximum value of the function is 6, and the minimum value is 2. ✓

**Problem 10.** This is a straightforward application of the Chain Rule. First compute $z_t$ in terms of $y = y(s, t)$:

$$z_t = z_x x_t + z_y y_t = y \cdot s^2 + (x - 3y^2)y_t = ys^2 + (s^2t - 3y^2)y_t.$$

Now plug in $(2, 1)$ for $(s, t)$, and use the information given:

$$z_t(2, 1) = y(2, 1) \cdot 4 + (4 \cdot 1 - 3(y(2, 1))^2)y_t(2, 1) = -4 - 4 + 3 = \boxed{-5}.\ ✓$$

**Problem 11.**

(a) The gradient points in the direction of greatest increase, so we compute the gradient:

$$\nabla T = \left\langle e^{x-y}y^2, -e^{x-y}y^2 + 2e^{x-y}y \right\rangle.$$

Plug in the point $(1, 1)$ to get

$$\nabla T(1, 1) = \langle 1, -1 + 2 \rangle = \langle 1, 1 \rangle.$$

So the bumblebee should move in direction $\langle 1, 1 \rangle$ to get warmer the fastest. ✓

The fastest rate of increase (i.e., the maximum value of the directional derivative in any direction) is the directional derivative in the direction of the gradient, which is the length of the gradient. In this case, the length of the gradient is $\sqrt{1^2+1^2} = \sqrt{2}$, so the fastest rate of increase of the temperature is $\sqrt{2}$. ✓

(b) Compute the directional derivative: $D_{\langle 0,1\rangle}T = \nabla T \cdot \langle 0,1\rangle = 1$. Since the directional derivative is positive in this direction, the temperature will increase. ✓

(c) Now we want to find vectors orthogonal to the graph of $T$ (not just orthogonal to the level curves), so we should rewrite our surface as a level surface and compute the gradient: $z = T(x,y) = 1$ rearranges to become $e^{x-y}y^2 - z - 1 = 0$. The gradient is

$$\left\langle e^{x-y}y^2, -e^{x-y}y^2 + 2e^{x-y}y, -1 \right\rangle = \langle 1, 1, -1\rangle .$$

The length of this vector is $\sqrt{3}$, so our two unit vectors are

$$\left\langle \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}} \right\rangle \text{ and } \left\langle -\tfrac{1}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}} \right\rangle . ✓$$

**Problem 12.**

(a) We look at the partials and determine where they are 0 or undefined.

$$\nabla f = \langle 2x - 4y, 2y - 4x\rangle .$$

Setting $2x - 4y = 2y - 4x = 0$ and solving for $x$ and $y$ gives $x = y = 0$. To classify this critical point, we use the second derivative test:

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & -4 \\ -4 & 2 \end{vmatrix} = -12 < 0.$$

So there is a saddle point at the origin.

(b) If $f$ is a continuous function defined on a closed and bounded domain $D$, then $f$ takes a maximum and minimum value on $D$. It achieves its extreme values either at critical points of $f$ or on the boundary of $D$.

(c) Since $f$ has a saddle point at the origin (and no other critical points), we know that $f$ will attain its extreme values on the boundary of the domain. The four pieces of the boundary are parametrized as follows:

$$\begin{aligned} y = x+1 \quad & x \in [-1,0]; \\ \langle \cos t, \sin t\rangle \quad & t \in [0, \pi/2]; \\ y = x-1 \quad & x \in [-1,1] \\ x = -1 \quad & y \in [-2,0]. \end{aligned}$$

We might as well record now the values of $f$ at the 'corners':

$$f(-1,0) = 1$$
$$f(0,1) = 1$$
$$f(1,0) = 1$$
$$f(-1,-2) = -3.$$

First, we consider the vertical line $x = -1$.

$$f(-1,y) = 1 + y^2 + 4y = y^2 + 4y + 1$$

has derivative $2y + 4$, and so its only critical point is $y = -2$, which we've already accounted for.

Now, consider the topmost diagonal line, $y = x + 1$. The function on that line looks like

$$f(x, x+1) = x^2 + (x+1)^2 - 4x(x+1) = -3x^2 + (x+1)^2 - 4x,$$

which has derivative $-4x - 2$. So the only critical point is $x = -\frac{1}{2}$, which gives the possible extreme value

$$f(-\tfrac{1}{2}, \tfrac{1}{2}) = \tfrac{3}{2}.$$

Next, consider the circle $\langle \cos t, \sin t\rangle$, $t \in [0, \pi/2]$. Restricted to this portion of the domain, the function looks like

$$f(\cos t, \sin t) = 1 - 4\cos t \sin t = 1 - 2\sin(2t),$$

which has derivative $-4\cos t$. So the only critical point here is $t = \pi/4$, which gives the possible extreme value

$$f(\cos\tfrac{\pi}{4}, \sin\tfrac{\pi}{4}) = f(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}) = \tfrac{1}{2} + \tfrac{1}{2} - 4(\tfrac{1}{2}) = -1.$$

Finally, consider the bottom diagonal line $y = x - 1$. Restricted to this portion of the domain, the function looks like

$$f(x, x-1) = x^2 + (x-1)^2 - 4x(x-1)$$

which has derivative $8x - 2 - 4x^2$. Setting this equal to 0 gives the equation $2x^2 - 4x + 1$, which has solutions $x = 1 \pm \frac{1}{\sqrt{2}}$, by the quadratic formula. Since $x = 1 + \frac{1}{\sqrt{2}}$ is outside the domain $[-1, 1]$, we only have to consider $x = 1 - \frac{1}{\sqrt{2}}$. We have

$$f(1 - \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}) = (1 - \tfrac{1}{\sqrt{2}})^2 + \tfrac{1}{2} - 4(1 - \tfrac{1}{\sqrt{2}})(\tfrac{1}{\sqrt{2}}) = 4 - 3\sqrt{2} \approx -0.2426.$$

So the absolute max is $f(-\frac{1}{2}, \frac{1}{2}) = \frac{3}{2}$, and the absolute min is $f(-1,-2) = -3$. ✓