RECURRENCE RELATION EXAMPLE

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This is a detailed explanation of a recurrence relation example I started working out in discussion on Thursday, 6 Feb.

Here is the recurrence:

$$t_0 = 0, \quad t_1 = 1 \tag{$*$}$$

$$t_{n+2} = 2t_n + t_{n+1}. \tag{$**$}$$

First we solve by guessing that $t_n = r^n$:

$$r^{n+2} = 2r^n + r^{n+1}.$$

Divide each side by r^n and rearrange to get $r^2 - r - 2 = 0$, which has solutions $r = 2$ and $r = -1$. The solution should be of the form

$$a2^n + b(-1)^n,$$

so now we need to use the initial conditions to solve for a and b.

$$0 = a2^0 + b(-1)^0 = a + b,$$
$$1 = a2^1 + b(-1)^1 = 2a - b.$$

Add these two equations to get $3a = 1$ and $b = -a$, so $a = \frac{1}{3}$ and $b = -\frac{1}{3}$. Our solution should be

$$\frac{1}{3}2^n - \frac{1}{3}(-1)^n.$$

Let's prove by induction that this is the solution. That is, let's prove that if s_n solves the recurrence ($**$) and the initial conditions ($*$), then for every $n \in \mathbb{N}$, $s_n = \frac{1}{3}2^n - \frac{1}{3}(-1)^n$.

Base case:

$$\tfrac{1}{3}2^0 - \tfrac{1}{3}(-1)^0 = \tfrac{1}{3} - \tfrac{1}{3} = 0 = s_0. \checkmark$$
$$\tfrac{1}{3}2^1 - \tfrac{1}{3}(-1)^1 = \tfrac{2}{3} + \tfrac{1}{3} = 1 = s_1. \checkmark$$

Inductive step:

Suppose inductively that $s_m = \frac{1}{3}2^m - \frac{1}{3}(-1)^m$ for all $m < n$. (This is what's sometimes called 'strong induction'.)

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In particular,

$$s_{n-2} = \tfrac{1}{3}2^{n-2} - \tfrac{1}{3}(-1)^{n-2}$$
$$s_{n-1} = \tfrac{1}{3}2^{n-1} - \tfrac{1}{3}(-1)^{n-1}.$$

We use these two facts and the recurrence and then do some algebraic manipulation to prove that $s_n = \frac{1}{3}2^n - \frac{1}{3}(-1)^n$.

$$\begin{aligned} s_n &= 2s_{n-2} + s_{n-1} \\ &= 2(\tfrac{1}{3}2^{n-2} - \tfrac{1}{3}(-1)^{n-2}) + (\tfrac{1}{3}2^{n-1} - \tfrac{1}{3}(-1)^{n-1}) \\ &= \tfrac{2}{3}2^{n-2} + \tfrac{1}{3}2^{n-1} - \tfrac{2}{3}(-1)^{n-2} - \tfrac{1}{3}(-1)^{n-1} \\ &= \tfrac{1}{3}2^{n-1} + \tfrac{1}{3}2^{n-1} - \tfrac{2}{3}(-1)^{n-2} + \tfrac{1}{3}(-1)^{n-2} \\ &= \tfrac{2}{3}2^{n-1} - \tfrac{1}{3}(-1)^{n-2} \\ &= \tfrac{1}{3}2^n - \tfrac{1}{3}(-1)^n, \end{aligned}$$

as desired. ✓