Math 32A Final Review (Fall 2012)

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Contents

Introduction

These notes serve as a review of the major topics covered in the multivariable differential calculus class taught at UCLA. They are intended to offer a condensed summary of the basic definitions and results we saw throughout the course. Emphasis is given to example problems. This review is not meant to be comprehensive, but gives an outline of a substantial portion of the material covered during the course.

The material is broken into three sections—geometry of $\mathbf{R}^3$, vector valued functions, and functions of several variables—which correspond to the three major themes of the course. The first section deals with notions of distance, vectors, angles, lines and planes, as well as quadric surfaces. These concepts give us the algebraic and geometric tools required to do calculus in $\mathbf{R}^3$ in the sequel.

The second section introduces vector valued functions (of a single variable) and parametrized curves. Classically, these concepts correspond to the trajectories of physical particles. We generalize the notion of the derivative to vector valued functions. This allows us to connect the algebraic aspects of these functions to geometry, which also lends a physical interpretation to a function and its derivatives.

Finally, the third section considers functions of several variables. Again, we are able to generalize the notion of derivative to this context in the form of partial derivatives and the gradient. Partial derivatives and gradients give us information about the local behavior of functions. In particular, partial derivatives give a criterion for when a function can achieve maximal and minimal values. This allows us to introduce optimization—an enormous and rich field in its own right. The material culminates with the method of Lagrange multipliers for optimization with constraints.

1 Geometry of $\mathbf{R}^3$

1.1 **definitions and results** The set $\mathbf{R}^3$ represents three dimensional space. There is a distinguished point which we called the origin (denoted $\mathcal{O}$) and all points are defined relative to the origin. We represent a point P in $\mathbf{R}^3$ by a triple of real numbers $P = (x, y, z)$ called the coordinates of P. The origin has coordinates $\mathcal{O} = (0, 0, 0)$. Given two points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ in $\mathbf{R}^3$, the ***distance*** from P_1 to P_2 is defined by

$$\text{Dist}(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \tag{1}$$

We also consider ***vectors*** in $\mathbf{R}^3$, which we think of as arrows—quantities with magnitude and direction. Vectors are also defined by three coordinates, but to distinguish them from points, we use angle brackets: $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. Given points P and Q as before, the vector pointing from P to Q has coordinates

$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle . \tag{2}$$

Given a vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, we define its ***length*** or ***magnitude*** to be

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}. \tag{3}$$

We call a vector $\mathbf{u}$ a ***unit vector*** if $\|\mathbf{u}\| = 1$.

There are two operations on vectors: ***addition*** and ***scalar multiplication***. Let $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ be vectors and $c \in \mathbf{R}$ a scalar. Then we define

$$\mathbf{v} + \mathbf{w} = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle \quad \text{and} \quad c\mathbf{v} = \langle cv_1, cv_2, cv_3 \rangle . \tag{4}$$

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ with $\mathbf{v}$, $\mathbf{u}$ and c as before. Then vector addition and scalar multiplication obey the following properties:

$$\begin{aligned} &\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \\ &(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}) \\ &c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v} \end{aligned} \tag{5}$$

Additionally, we define the ***dot product*** and ***cross product*** of two vectors by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \tag{6}$$

and

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle \tag{7}$$

respectively. Notice that the dot product is a scalar while the cross product is a vector. We can interpret the dot product geometrically via the equation

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta \tag{8}$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. We say that $\mathbf{u}$ and $\mathbf{v}$ are ***orthogonal*** (or perpendicular) if $\mathbf{u} \cdot \mathbf{v} = 0$; that is, the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\pi/2$. The cross product is characterized by the following properties:

1. $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$,
2. $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin\theta$,
3. the direction of $\mathbf{u} \times \mathbf{v}$ is determined by the right-hand rule.

A ***line*** in $\mathbf{R}^3$ is a set of the form

$$L = \{\mathbf{r}_0 + t\mathbf{v} \mid t \in \mathbf{R}\}. \tag{9}$$

The equation $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$ is the ***vector equation*** of the line L. Equivalently, we can consider the ***parametric equation*** of L

$$\begin{aligned} x(t) &= at + x_0 \\ y(t) &= bt + y_0 \\ z(t) &= ct + z_0 \end{aligned} \tag{10}$$

where $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ and $\mathbf{v} = \langle a, b, c \rangle$. Notice that a line is defined by a point on the line ($\mathbf{r}_0$) and a direction ($\mathbf{v}$).

A ***plane*** in $\mathbf{R}^3$ can be expressed in the form

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = 0 \quad \text{where} \quad \mathbf{x} = \langle x, y, z \rangle \tag{11}$$

The vector $\mathbf{n}$ is a ***normal vector*** for the plane. Similar to lines, a plane is defined by two vectors: a normal vector ($\mathbf{n}$) and a point on the plane ($\mathbf{x}_0$). Equivalently, a plane can be defined by the equation

$$ax + by + cz = d \quad \text{where} \quad \mathbf{n} = \langle a, b, c \rangle \quad \text{and} \quad \mathbf{n} \cdot \mathbf{x}_0 = d. \tag{12}$$

Finally we recall the standard quadric surfaces in $\mathbf{R}^3$:

- Ellipsoid: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$
- Hyperboloid of 1 sheet: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{z}{c}\right)^2 + 1$
- Hyperboloid of 2 sheets: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{z}{c}\right)^2 - 1$
- Elliptic Paraboloid: $z = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$
- Hyperbolic Paraboloid: $z = \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2$
- Elliptic Cone: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{z}{c}\right)^2$

1.2 problems

Example 1 (Intersection of planes)**.** Consider the planes given by

$$x + 2y + 3z = 6 \quad \text{and} \quad 2x - y - z = 0.$$

(a) Find the angle between the two planes

(b) Find the parametric equation of the line representing the intersection of the two planes (Hint: $(1, 1, 1)$ lies in both planes)

(c) Find the equation of plane perpendicular to both planes going through the point $(1, 1, 1)$.

Solution. For part (a), recall that the angle between the two planes is the same as the angle between their normal vectors. In this case, the normal vectors of the two planes are

$$\mathbf{n}_1 = \langle 1, 2, 3 \rangle \quad \text{and} \quad \mathbf{n}_2 = \quad \langle 2, -1, -1 \rangle$$

respectively. We compute the angle between the vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ using the formula

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = \|\mathbf{n}_1\| \, \|\mathbf{n}_2\| \cos\theta$$

where θ is the angle between the normal vectors. We find

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = (1)(2) + (2)(-1) + (3)(-1) = -3$$

while

$$\|\mathbf{n}_1\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14} \quad \text{and} \quad \|\mathbf{n}_2\| = \sqrt{2^2 + (-1)^2 + (-1)^2} = \sqrt{6}.$$

Therefore,

$$\theta = \cos^{-1}\left(\frac{-3}{\sqrt{6}\sqrt{14}}\right) = \cos^{-1}\left(\frac{-3}{2\sqrt{21}}\right).$$

For part (b), we must find the equation of the line of intersection of the two planes. Recall that in order to define a line, we must find a point on the line and the direction vector $\mathbf{v}$ in which the line points. In the hint, we are given that the point $(1, 1, 1)$ lies on both planes, hence lies on the line representing their intersection. Further, the line of intersection is perpendicular to both normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, as it lies lies in both planes. Therefore, we can take the direction vector $\mathbf{v}$ to be $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2$. We compute

$$\begin{aligned}
\mathbf{v} &= \mathbf{n}_1 \times \mathbf{n}_2 \\
&= \langle 1, 2, 3 \rangle \times \langle 2, -1, -1 \rangle \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 2 & -1 & -1 \end{vmatrix} \\
&= \mathbf{i}(-2 + 3) - \mathbf{j}(-1 - 6) + \mathbf{k}(-1 - 4) \\
&= \langle 1, 7 - 5 \rangle.
\end{aligned}$$

Then the equation of the line is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = \langle 1, 1, 1 \rangle + t\,\langle 1, 7, -5 \rangle = \langle 1 + t, 1 + 7t, 1 - 5t \rangle.$$

Finally, for part (c), we recall that the equation of a plane with normal vector $\mathbf{n}$ going through the point $\mathbf{r}_0$ is given by

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0.$$

As in part (b), $\mathbf{r}_0 = \langle 1, 1, 1 \rangle$. Further, the normal vector $\mathbf{n}$ of the new plane is perpendicular to the normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ of the first two planes. Therefore, we may take $\mathbf{n} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, 7, -5 \rangle$, as calculated for part (b). Therefore, the equation of the plane is

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \implies \langle 1, 7, -5 \rangle \cdot \langle x - 1, y - 1, z - 1 \rangle = 0 \implies x + 7y - 5z = 3.$$

$\square$

Example 2 (Quadric surfaces)**.** Classify the quadric surface given by

$$x^2 - y^2 - z^2 - 2x - 4y + 6z = 12$$

and classify its horizontal traces ($z =$ const).

Solution. Completing the square gives

$$(x-1)^2 - (y+2)^2 - (z-3)^2 = 0$$

which is the equation of an elliptic cone opening in the x direction with center $(1, -2, 3)$. The horizontal traces are given by

$$(x-1)^2 - (y+2)^2 = (k-3)^2,$$

which for $k \neq 3$ is the equation of a hyperbola with center $(1, -2)$. For $k = 3$, the trace is a pair of intersecting lines. □

2 Vector Valued Functions

2.1 definitions and results

A ***vector-valued function*** is a function $\mathbf{r} : I \to \mathbf{R}^3$ where $I \subset \mathbf{R}$ is an interval. We can write such a function in terms of its coordinates as

$$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle . \tag{13}$$

We define limits, derivatives and integrals component-wise:

$$\begin{aligned} &\lim_{t\to t_0} \mathbf{r}(t) = \langle \lim_{t\to t_0} x(t), \lim_{t\to t_0} y(t), \lim_{t\to t_0} z(t)\rangle \\ &\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle \\ &\int_{t_0}^t \mathbf{r}(u)\,du = \left\langle \int_{t_0}^t x(u)\,du, \int_{t_0}^t y(u)\,du, \int_{t_0}^t z(u)\,du \right\rangle \end{aligned} \tag{14}$$

The derivative $\mathbf{r}'(t_0)$ (if it exists) is the ***tangent vector*** to $\mathbf{r}(t)$ at $t = t_0$. The ***arc-length*** of $\mathbf{r}(t)$ for $a \le t \le b$ is given by

$$s(t) = \int_a^t \|\mathbf{r}(u)\|\ du = \int_a^t \sqrt{(x'(u))^2 + (y'(u))^2 + (z'(u))^2}\,du. \tag{15}$$

The function $\mathbf{r}(t)$ is said to be an ***arc-length parametrization*** if

$$s(t) = t \quad \text{or equivalently} \quad \|\mathbf{r}(t)\| = 1 \quad \text{for all} \quad t. \tag{16}$$

We define the ***unit tangent***, ***normal*** and ***binormal*** vectors by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}, \qquad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}, \qquad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) \tag{17}$$

respectively. The plane containing $\mathbf{r}(t_0)$ which is parallel to $\mathbf{T}(t_0)$ and $\mathbf{N}(t_0)$ is the ***osculating plane***; the plane containing $\mathbf{r}(t_0)$ which is perpendicular to $\mathbf{T}(t_0)$ is the ***normal plane***. The ***curvature*** of $\mathbf{r}(t)$ is given by

$$\kappa(t) = \left\| \frac{d\mathbf{T}}{ds} \right\| = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}. \tag{18}$$

Finally, we offer a physical interpretation of $\mathbf{r}(t)$ and its derivatives. Suppose the trajectory of a particle is given by $\mathbf{r}(t)$. Then $\mathbf{v}(t) = \mathbf{r}'(t)$ is the ***velocity*** of the particle, $v(t) = \|\mathbf{v}(t)\|$ is its ***speed***, and $\mathbf{a}(t) = \mathbf{r}''(t)$ is its ***acceleration***.

2.2 problems

Example 3 (Angle of intersection)**.** Consider the two curves in $\mathbf{R}^3$ given by

$$\mathbf{r}_1(t) = \langle \cos t, \sin t, t\rangle \quad \text{and} \quad \mathbf{r}_2(t) = \left\langle t, (t-1)^2, (t-1)^3 \right\rangle .$$

Notice that the two curves intersect at the point $(1, 0, 0)$. Find the angle of intersection of the two curves at that point.

Solution. The angle of intersection of the two curves at the point $(1, 0, 0)$ is the angle between their tangent vectors at that point. First notice that

$$\langle 1, 0, 0\rangle = \mathbf{r}_1(0) = \mathbf{r}_2(1)$$

so we must evaluate $\mathbf{r}_1'$ at 0 and $\mathbf{r}_2'$ at 1 to find the tangent vectors at the point $(1, 0, 0)$. We compute

$$\mathbf{r}_1'(0) = \langle -\sin t, \cos t, 1\rangle \Big| t = 0 = \langle 0, 1, 1\rangle$$

and

$$\mathbf{r}_2'(1) = \left\langle 1, 2(t-1), 3(t-1)^2\right\rangle \Big| t = 1 = \langle 1, 0, 0\rangle\,.$$

Recall that in general, the angle θ between two vectors $\mathbf{u}$ and $\mathbf{v}$ satisfies $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$. Therefore, we compute the dot product

$$\mathbf{r}_1'(0)\cdot\mathbf{r}_2'(1) = \langle 0, 1, 1\rangle\cdot\langle 1, 0, 0\rangle = 0$$

implying that the angle of intersection, θ is $\theta = \pi/2$. ☐

Example 4 (Motion on a sphere)**.** Suppose $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ are curves in $\mathbf{R}^3$.

(a) Prove the product formula

$$\frac{d}{dt}(\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)) = \mathbf{r}_1'(t)\cdot\mathbf{r}_2(t) + \mathbf{r}_1(t)\cdot\mathbf{r}_2'(t).$$

(b) Suppose $\mathbf{r}(t)$ lies on a sphere of radius k centered at the origin (that is $\|\mathbf{r}(t)\| = k$ for all t). Use part (a) to prove that $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$ for all t.

Solution. For part (a), write $\mathbf{r}_1$ and $\mathbf{r}_2$ in terms of their components

$$\mathbf{r}_1(t) = \langle x_1(t), y_1(t), z_1(t)\rangle\,, \quad \text{and} \quad \mathbf{r}_2(t) = \langle x_2(t), y_2(t), z_2(t)\rangle\,.$$

Then we can write

$$\begin{aligned}
\frac{d}{dt}(\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)) &= \frac{d}{dt}(x_1(t)x_2(t) + y_1(t)y_2(t) + z_1(t)z_2(t)) \\
&= x_1'(t)x_2(t) + x_1(t)x_2'(t) + y_1'(t)y_2(t) + y_1(t)y_2'(t) \\
&\quad + z_1'(t)z_2(t) + z_1(t)z_2'(t) \\
&= \langle x_1'(t), y_1'(t), z_1'(t)\rangle\cdot\langle x_2(t), y_2(t), z_2(t)\rangle \\
&\quad + \langle x_1(t), y_1(t), z_1(t)\rangle\cdot\langle x_2'(t), y_2'(t), z_2'(t)\rangle \\
&= \mathbf{r}_1'(t)\cdot\mathbf{r}_2(t) + \mathbf{r}_1(t)\cdot\mathbf{r}_2'(t)
\end{aligned}$$

which is what we wanted to show.

For part (b), we can can write the condition that $\mathbf{r}(t)$ is constrained to the sphere of radius k centered at the origin as

$$\mathbf{r}(t)\cdot\mathbf{r}(t) = k^2.$$

Differentiating both sides of the equation with respect to t (where k is constant) and applying the conclusion of part (a) gives

$$\mathbf{r}'(t)\cdot\mathbf{r}(t) + \mathbf{r}(t)\cdot\mathbf{r}'(t) = 0.$$

Since the dot product is commutative, this implies

$$2\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$$

hence $\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$, as desired. □

Example 5 (Computing curvature)**.** Consider the curve given by

$$\mathbf{r}(t) = \left\langle \frac{1}{3}t^3, \frac{1}{\sqrt{2}}t^2, t \right\rangle.$$

Find the unit tangent vector $\mathbf{T}(t)$ of $\mathbf{r}(t)$ and compute the curvature $\kappa(t)$ of $\mathbf{r}$.

Solution. Recall that the unit tangent vector is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}.$$

So we compute

$$\mathbf{r}'(t) = \left\langle t^2, \sqrt{2}t, 1 \right\rangle.$$

Therefore

$$\|\mathbf{r}'(t)\| = \sqrt{(t^2)^2 + 2t^2 + 1} = \sqrt{(t^2+1)^2} = t^2 + 1.$$

Hence

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \left\langle \frac{t^2}{t^2+1}, \frac{t\sqrt{2}}{t^2+1}, \frac{1}{t^2+1} \right\rangle.$$

We compute the curvature (as we usually do) using the formula

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

We've already computed $\mathbf{r}'(t) = \langle t^2, t\sqrt{2}, 1 \rangle$, so

$$\mathbf{r}''(t) = \left\langle 2t, \sqrt{2}, 0 \right\rangle.$$

We compute the cross product

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t^2 & t\sqrt{2} & 1 \\ 2t & \sqrt{2} & 0 \end{vmatrix} = \left\langle -\sqrt{2}, 2t, -\sqrt{2}t^2 \right\rangle.$$

Then

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \sqrt{(-\sqrt{2})^2 + (2t)^2 + (t^2\sqrt{2})^2} = \sqrt{2}(1+t^2).$$

Therefore,

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\sqrt{2}}{(1+t^2)^2}.$$

□

Example 6 (Acceleration and curvature)**.** Suppose $\mathbf{r}(t)$ describes the motion of a particle in $\mathbf{R}^3$.

(a) Show that the acceleration $\mathbf{a}(t) = \mathbf{r}''(t)$ can be written as

$$\mathbf{a}(t) = v'(t)\mathbf{T}(t) + (v(t))^2\kappa(t)\mathbf{N}(t)$$

where $v(t) = \|\mathbf{r}'(t)\|$ and $\kappa(t) = \left\|\frac{dT}{ds}\right\|$ is the curvature of the trajectory. (Hint: write $\mathbf{v}(t) = v(t)\mathbf{T}(t)$ and differentiate with respect to t.)

(b) The motion of particle a particle accelerator is approximated by the equation

$$\mathbf{r}(t) = \left\langle R\cos\left(\frac{1}{2}at^2\right), R\sin\left(\frac{1}{2}at^2\right), 0\right\rangle.$$

Use part (a) to find the tangential and normal components of the particles acceleration as a function of t.

Solution. For part (a), we apply the hint writing $\mathbf{v}(t) = v(t)\mathbf{T}(t)$. Then we can write the acceleration as

$$\begin{aligned}
\mathbf{a}(t) &= \frac{d}{dt}\mathbf{v}(t) \\
&= \frac{d}{dt}(v(t)\mathbf{T}(t)) \\
&= v'(t)\mathbf{T}(t) + v(t)\frac{d}{dt}\mathbf{T}(t) && \text{(product rule)} \\
&= v'(t)\mathbf{T}(t) + v(t)\frac{d\mathbf{T}}{ds}\frac{ds}{dt} && \text{(chain rule)} \\
&= v'(t)\mathbf{T}(t) + (v(t))^2\frac{d\mathbf{T}}{ds} && (ds/dt = v(t)) \\
&= v'(t)\mathbf{T}(t) + (v(t))^2\left\|\frac{d\mathbf{T}}{ds}\right\|\mathbf{N}(t) && \text{(def. of } \mathbf{N}) \\
&= v'(t)\mathbf{T}(t) + (v(t))^2\kappa(t)\mathbf{N}(t). && \text{(def. of } \kappa)
\end{aligned}$$

which is the desired result.

For part (b), compute

$$\mathbf{v}(t) = \mathbf{r}'(t) = Rat\left\langle -\sin\left(\frac{1}{2}at^2\right), \cos\left(\frac{1}{2}at^2\right), 0\right\rangle$$

so that

$$v(t) = \|\mathbf{v}(t)\| = Rat.$$

Therefore, the tangential component of acceleration is

$$v'(t) = Ra.$$

To obtain the the normal component of acceleration, notice that $\mathbf{r}(t)$ is a parametrization of a circle of radius R. Since κ is the reciprocal of the radius of the osculating circle, we get (without any computation) that $\kappa(t) = 1/R$ for all t (you should confirm this with direct computation). Therefore, the normal component of acceleration is

$$(v(t))^2\kappa(t) = (Rat)^2/R = Ra^2t^2.$$

□

3 Functions of Several Variables

3.1 **definitions and results** In this section, we consider functions $f : \mathbf{R}^n \to \mathbf{R}$ for $n = 2$ and 3. Suppose f is a function of 2 variables defined on a domain D containing points arbitrarily close to a point (a, b). Then

$$\lim_{(x,y)\to(a,b)} f(x, y) = L \tag{19}$$

if for every $\varepsilon > 0$, there exists a corresponding $\delta > 0$ such that if $(x, y) \in D$ and

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \tag{20}$$

then

$$|f(x, y) - L| < \varepsilon. \tag{21}$$

The function f is said to be ***continuous*** at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b). \tag{22}$$

An important tool for computing limits is the ***squeeze theorem***: Suppose f, g and h are functions of two variables and that

$$g(x, y) \leq f(x, y) \leq h(x, y) \tag{23}$$

for all (x, y) in some disk containing (a, b) and that

$$\lim_{(x,y)\to(a,b)} g(x, y) = \lim_{(x,y)\to(a,b)} g(x, y) = L. \tag{24}$$

Then

$$\lim_{(x,y)\to(a,b)} f(x, y) = L. \tag{25}$$

To show that $\lim_{(x,y)\to(a,b)} f(x, y)$ ***does not exist*** it suffices to find two continuous curves $\mathbf{r}_1(t) = \langle x_1(t), y_1(t)\rangle$ and $\mathbf{r}_2(t) = \langle x_2(t), y_2(t)\rangle$ such that $\mathbf{r}_1(0) = \mathbf{r}_2(0) = \langle a, b\rangle$, but for which

$$\lim_{t\to 0} f(x_1(t), y_1(t)) \neq \lim_{t\to 0} f(x_2(t), y_2(t)). \tag{26}$$

Let f be a function of 2 variables. Then we define the ***partial derivatives*** of f at (a, b) (if they exist) to be

$$\frac{\partial f}{\partial x}(a, b) = f_x(a, b) = \lim_{h\to 0} \frac{f(a+h, b) - f(a, b)}{h} \tag{27}$$

and

$$\frac{\partial f}{\partial y}(a, b) = f_y(a, b) = \lim_{h\to 0} \frac{f(a, b+h) - f(a, b)}{h}. \tag{28}$$

We can take higher-order derivatives. A useful result is ***Clairaut's theorem***: if f_{xy} and f_{yx} are both defined and continuous on a disk containing the point (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b). \tag{29}$$

If $\mathbf{u} = \langle u, v\rangle$ is a unit vector, then we define the ***directional derivative in the direction of*** $\mathbf{u}$ to be

$$D_{\mathbf{u}} f(x, y) = \lim_{h\to 0} \frac{f(x+uh, y+vh) - f(x, y)}{h}. \tag{30}$$

The ***gradient*** of f at (a, b) is defined by

$$\nabla f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle . \tag{31}$$

The gradient has the following geometric interpretation: ∇f points in the direction of greatest increase of f and $\|\nabla f\|$ is the slope of f in that direction. Assuming that f is differentiable at the point (a, b) (see below) we can compute

$$D_{\mathbf{u}} f(a,b) = (\nabla f(a,b)) \cdot \mathbf{u}. \tag{32}$$

Given a ***level set*** of a function, that is set S of the form $f(x, y, z) = k$ for some constant k, and a point (a, b, c) satisfying $f(a, b, c) = k$, the gradient $\nabla f(a, b, c)$ is a normal vector to S at (a, b, c).

Given a point (a, b) we define the ***linearization*** of f at (a, b) to be

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b). \tag{33}$$

We say that f is ***differentiable at*** (a, b) if we can write

$$f(x,y) = L(x,y) + e(x,y) \quad \text{where} \quad \lim_{(x,y)\to(a,b)} \frac{e(x,y)}{\sqrt{(x-a)^2 + (y-b)^2}} = 0. \tag{34}$$

In order to check that f is differentiable at (a, b), it suffices to check that f_x and f_y are both defined and continuous on some disk D containing (a, b).

If $f(x_1, x_2, \ldots, x_n)$ is a function of n variables, and each x_i is a function of m variables, $x_i = x_i(t_1, t_2, \ldots, t_m)$, then we can apply the ***chain rule*** to compute partial derivatives of f with respect to t_k:

$$\frac{\partial f}{\partial t_k} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_k} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_k} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_k} \tag{35}$$

A function f of two variables has a ***local maximum*** (respectively ***local minimum***) at (a, b) if $f(x, y) \le f(a, b)$ (respectively $f(a, b) \le f(x, y)$) for all (x, y) in a sufficiently small disk with center (a, b). We say that (a, b) is a ***critical point*** for f if the following two conditions hold:

1. $f_x(a, b) = 0$ or $f_x(a, b)$ does not exist
2. $f_y(a, b) = 0$ or $f_y(a, b)$ does not exist.

An important result is that all (interior) local maxima and minima occur at critical points. The ***second derivative test*** allows us to classify some critical points of f: let (a, b) be a critical point of f, and define

$$D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - (f_{xy}(x,y))^2. \tag{36}$$

The we can draw the following conclusions:

- if $D(a, b) > 0$ and $f_{xx}(a, b) < 0$ then $f(a, b)$ is a local maximum
- if $D(a, b) > 0$ and $f_{xx}(a, b) > 0$ then $f(a, b)$ is a local minimum
- if $D(a, b) < 0$ then (a, b) is a ***saddle point*** for f.

If $D(a, b) = 0$ we cannot draw any conclusions about the behavior of f at (a, b).

Often, we would like to find maxima and minima of a function $f(x, y, z)$ subject to a constraint of the form $g(x, y, z) = k_1$ for some constant k. To this end, we apply the method of ***Lagrange Multipliers***. In this context, we have a different criterion for when a point (a, b, c) satisfying $g(a, b, c) = k$ can be a maximum or minimum value for f:

$$\nabla f(a, b, c) = \lambda \nabla g(a, b, c) \tag{37}$$

for some scalar $\lambda \in \mathbf{R}$. If we have the additional constraint that $h(a, b, c) = k_2$, then we seek points (a, b, c) satisfying

$$\nabla f(a, b, c) = \lambda \nabla g(a, b, c) + \mu \nabla h(a, b, c) \tag{38}$$

for some scalars λ and μ.

3.2 problems

Example 7 (Limits)**.** Determine whether or not the following limits exist and prove your answer:

(a) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$;

(b) $\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^2+y^2}$.

Solution. (a) First, consider $f = xy/(x^2 + y^2)$ restricted to the x-axis, i.e., $y = 0$. Then

$$f(x, 0) = \frac{x \cdot 0}{x^2 + 0^2} = 0.$$

Therefore, $\lim_{x\to 0} f(x, 0) = 0$, so if the limit in (a) exists, it must be equal to 0. To show that the limit doesn't exist, it suffices to find another curve in the plane where the limit of f as $(x, y) \to (0, 0)$ along the curve is not equal to 0. Consider the curve given by $x = y$. Then we can write

$$f(x, x) = \frac{xx}{x^2 + x^2} = \frac{x^2}{2x^2}.$$

Then $\lim_{x\to 0} f(x, x) = 1/2 \neq 0$, so the limit does not exist.

(b) Let $g(x, y) = 2x^2y/(x^2 + y^2)$. As in part (a), if the limit of g as (x, y) approaches $(0, 0)$ exists, then it must be 0 (again, just consider the values of g on the x axis, i.e., with $y = 0$). Notice that for any choice of $y = x^a$ or $x = y^b$ we have

$$\lim_{x\to 0} g(x, x^a) = 0 \quad \text{and} \quad \lim_{y\to 0} g(y^b, y) = 0.$$

So we suspect that the limit might exist, in which case we must have $L = \lim_{(x,y)\to(0,0)} g(x, y) = 0$. Let us prove it. We appeal to the squeeze theorem. First, split up the fraction as the product

$$\frac{2x^2y}{x^2 + y^2} = \frac{2x^2}{x^2 + y^2} y.$$

Notice that the first term in the product is non-negative and in fact we have

$$0 \leq \frac{2x^2}{x^2 + y^2} \leq \frac{2x^2}{x^2} = 2$$

because $x^2 + y^2 \geq x^2$ for all x and y. Therefore, we have

$$0 \leq \frac{2x^2}{x^2+y^2} y \leq 2y.$$

Since the limits of the right and left hand sides of this expression are zero as $(x, y) \to (0, 0)$, the squeeze theorem implies that

$$\lim_{(x,y)\to(0,0)} \frac{2x^2}{x^2+y^2} y = 0$$

as desired.

□

Example 8 (Equation of tangent plane)**.** Show that the equation of the tangent plane to the ellipsoid given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

at the point (x_0, y_0, z_0) can be written as

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1.$$

Solution. In order to find the equation of a plane, we must first find a normal vector to the plane. Recall that the normal vector to a level surface $f(x, y, z) = k$ at a point (x_0, y_0, z_0) is given by the gradient $\nabla f(x_0, y_0, z_0)$. In this case, we have $f(x, y, z) = x^2/a^2 + y^2/b^2 + z^2/c^2$. Therefore, we compute

$$\mathbf{n} = \nabla f(x_0, y_0, z_0) = \left\langle 2\frac{x_0}{a^2}, 2\frac{y_0}{b^2}, 2\frac{z_0}{c^2} \right\rangle.$$

The equation of a plane going through the point (x_0, y_0, z_0) with normal vector $\mathbf{n}$ is

$$\mathbf{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0.$$

In this case, we have

$$\begin{aligned}
&\left\langle 2\frac{x_0}{a^2}, 2\frac{y_0}{b^2}, 2\frac{z_0}{c^2} \right\rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0 \\
\implies\ & 2\frac{x_0}{a^2}(x - x_0) + 2\frac{y_0}{b^2}(y - y_0) + 2\frac{z_0}{c^2}(y - y_0) = 0 \\
\implies\ & \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} \\
\implies\ & \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} = 1.
\end{aligned}$$

The final implication holds because (x_0, y_0, z_0) is a point on the ellipsoid, hence its coordinates satisfy

$$\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} = 1.$$

□

Example 9 (Chain rule)**.** If $z = f(x, y)$ has continuous second-order partial derivatives and $x = r^2 + s^2$ and $y = 2rs$ find:

(a) $\frac{\partial z}{\partial r}$

(b) $\frac{\partial^2 z}{\partial r^2}$.

Solution. We apply the chain rule to compute

$$\begin{aligned}\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \\ &= \frac{\partial z}{\partial x}(2r) + \frac{\partial z}{\partial y}(2s) \\ &= 2\frac{\partial z}{\partial x}r + 2\frac{\partial z}{\partial y}s.\end{aligned}$$

Similarly, we employ the chain rule to compute

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} &= \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right) \\ &= \frac{\partial}{\partial r}\left(2\frac{\partial z}{\partial x}r + 2\frac{\partial z}{\partial y}s\right) \\ &= 2\left(\frac{\partial}{\partial r}\frac{\partial z}{\partial x}\right)r + 2\frac{\partial z}{\partial x}\left(\frac{\partial}{\partial r}r\right) + 2\left(\frac{\partial}{\partial r}\frac{\partial z}{\partial y}\right)s + 2\frac{\partial z}{\partial y}\left(\frac{\partial}{\partial r}s\right) \\ &= 2\left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y\partial x}\frac{\partial y}{\partial r}\right)r + 2\frac{\partial z}{\partial x}\cdot 1 + 2\left(\frac{\partial^2 z}{\partial x\partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r}\right)s + 2\frac{\partial z}{\partial y}\cdot 0 \\ &= 2\left(\frac{\partial^2 z}{\partial x^2}(2r) + \frac{\partial^2 z}{\partial y\partial x}(2s)\right)r + 2\frac{\partial z}{\partial x} + 2\left(\frac{\partial^2 z}{\partial x\partial y}(2r) + \frac{\partial^2 z}{\partial y^2}(2s)\right)s \\ &= 4r^2\frac{\partial^2 z}{\partial x^2} + 4s^2\frac{\partial^2 z}{\partial y^2} + 8rs\frac{\partial^2 z}{\partial x\partial y} + 2\frac{\partial z}{\partial x}.\end{aligned}$$

For the final equality, we applied Clairaut's theorem to the mixed partial derivatives. ☐

Example 10 (Wave equation)**.** Show that if f and g are twice differentiable functions of 1 variable and a is a constant, then

$$u(x,t) = f(x+at) + g(x-at)$$

is a solution to the wave equation $u_{tt} = a^2 u_{xx}$.

Solution. By the chain rule, we compute

$$\frac{\partial u}{\partial x} = f'(x+at)\frac{\partial}{\partial x}(x+at) + g'(x-at)\frac{\partial}{\partial x}(x-at) = f'(x+at) + g'(x-at).$$

Similarly,

$$\frac{\partial^2 u}{\partial x^2} = f''(x+at) + g''(x-at).$$

Meanwhile,

$$\frac{\partial u}{\partial t} = f'(x+at)\frac{\partial}{\partial t}(x+at) + g'(x-at)\frac{\partial}{\partial t}(x-at) = af'(x+at) - ag'(x-at)$$

and

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= af''(x+at)\frac{\partial}{\partial t}(x+at) - ag''(x-at)\frac{\partial}{\partial t}(x-at) \\ &= a^2 f''(x+at) + a^2 g''(x-at) \\ &= a^2(f''(x+at) + g''(x-at)) \\ &= a^2 \frac{\partial^2 u}{\partial x^2}\end{aligned}$$

which is what we wanted to show. □

Example 11 (Second derivative test)**.** Consider the function $f(x,y) = 3x - x^3 - 2y^2 + y^4$. Find all critical points and identify them as local minima, maximal or saddle points.

Solution. To find critical points, we must solve $f_x = 0$ and $f_y = 0$. We compute

$$f_x(x,y) = -3x^2 + 3, \qquad f_y(x,y) = 4y^3 - 4y.$$

Therefore

$$f_x(x,y) = 0 \implies -3x^2 + 3 = 0 \implies x = \pm 1$$

and

$$f_y(x,y) = 0 \implies 4y^3 - y = 0 \implies y(y+1)(y-1) = 0 \implies y = -1, 0, 1.$$

So the critical points are

$$\{(-1,-1), (-1,0), (-1,1), (1,-1), (1,0), (1,1)\}.$$

To identify these critical points, we employ the second derivative test. Recall that

$$D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - (f_{xy}(x,y))^2.$$

We compute the second derivatives as

$$f_{xx}(x,y) = -6x, \qquad f_{yy}(x,y) = 12y^2 - 4, \qquad f_{xy}(x,y) = 0$$

so that

$$D(x,y) = (-6x)(12y^2 - 4).$$

Then we evaluate $D(x,y)$ and $f_{xx}(x,y)$ at the critical points to identify min/max/saddle points:

(x,y)	$D(x,y)$	$f_{xx}(x,y)$	identification
$(1,-1)$	-48	-6	saddle
$(1,0)$	24	-6	local max
$(1,1)$	-48	-6	saddle
$(-1,-1)$	48	6	local min
$(-1,0)$	-24	6	saddle
$(-1,1)$	48	6	local min

□

Example 12 (Optimization on a bounded region)**.** Find the absolute maximum and minimum of the function $f(x, y) = x^2 y$ on the region defined by $x^2 + 2y^2 \leq 6$. (Hint: use Lagrange multipliers to identify max/min on the boundary.)

Solution. First, we identify critical points (if any) in the interior of the region, i.e., (x, y) with $x^2 + 2y^2 < 6$. We compute

$$f_x(x, y) = 2xy, \qquad f_y(x, y) = x^2.$$

These are both zero if and only if $x = 0$. If $x = 0$, we evaluate $f(0, y) = 0$, so all critical points on the interior of the region are of the form $(0, y)$ in which case $f(0, y) = 0$.

Now we turn our attention to the boundary of the region given by $x^2 + 2y^2 = 6$. We use the method of Lagrange multipliers with $f(x, y) = x^2 y$ and $g(x, y) = x^2 + 2y^2 = 6$. We compute

$$\nabla f(x, y) = \left\langle 2xy, x^2 \right\rangle, \quad \text{and} \quad \nabla g(x, y) = \langle 2x, 4y \rangle.$$

We can solve $\nabla f = \lambda \nabla g$ for λ:

$$\nabla f = \lambda \nabla g,\ g(x, y) = 6 \implies \begin{cases} 2xy = 2\lambda x \\ x^2 = 4\lambda y \\ x^2 + 2y^2 = 6. \end{cases}$$

Multiplying the first equation by x, the second equation by y and adding the resulting equations gives

$$2x^2 y + x^2 y = 2\lambda x^2 + 4\lambda y^2 = 2\lambda(x^2 + 2y^2) = 12\lambda.$$

Therefore, we must have

$$\lambda = \frac{1}{4} x^2 y.$$

Therefore, we have

$$2xy = \frac{1}{2} x^3 y, \qquad x^2 = x^2 y^2, \qquad x^2 + 2y^2 = 6.$$

All three equations are satisfied for

$$(x, y) = (0, \pm\sqrt{3}), \quad (\pm\sqrt{6}, 0), \quad (\pm 2, \pm 1),$$

so these are our boundary critical points. Plugging in all these points for f (including the interior critical points) we find that $f(\pm 2, 1) = 4$ and $f(\pm 2, -1) = -4$ are the absolute maximum and minimum for f on the region. $\square$

Example 13 (Lagrange with two constraints)**.** Let C be the intersection of the surfaces given by

$$z = xy \quad \text{and} \quad x^2 + y^2 = 8.$$

Find the points on C which are closest and furthest from the origin.

Solution. We can view this problem as optimizing the function

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$

(i.e., the distance from (x, y, z) to the origin) subject to the constraints

$$xy - z = 0 \quad \text{and} \quad x^2 + y^2 = 8.$$

Since the square-root in the definition of f is cumbersome, so we can do without it: finding the points (a, b, c) that minimize and maximize the square distance will also give minimum and maximum values for the distance. So we instead consider the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

and just take the square-root at the end. Let $g(x, y, z) = xy - z$ and $h(x, y, z) = x^2 + y^2$. Then we must solve the equations

$$\nabla f = \lambda \nabla g + \mu \nabla h. \qquad (*)$$

We compute the gradients

$$\begin{aligned} \nabla f(x, y, z) &= \langle 2x, 2y, 2z \rangle \\ \nabla g(x, y, z) &= \langle y, x, -1 \rangle \\ \nabla h(x, y, z) &= \langle 2x, 2y, 0 \rangle . \end{aligned}$$

Therefore, we must solve the simultaneous system of equations

$$\begin{aligned} 2x &= \lambda y + 2\mu x \\ 2y &= \lambda x + 2\mu y \\ 2z &= -\lambda \\ z &= xy \\ x^2 + y^2 &= 8. \end{aligned}$$

Using the third equation, we can get rid of λ by replacing it with $\lambda = -2z$. Plugging this into the first equations and dividing through by 2 gives

$$x + zy = \mu x \quad \text{and} \quad y + zx = \mu y.$$

Multiplying the first of these equations by y and the second by x gives

$$xy + zy^2 = xy + zx^2 \implies zx^2 = zy^2.$$

This equation gives two possibilities: either $z = 0$ or $x^2 = y^2$. If $z = 0$, then the first constraint equation gives that $x = 0$ or $y = 0$, hence

$$(x, y, z) = \left(\pm 2\sqrt{2}, 0, 0\right) \quad \text{or} \quad (0, \pm 2\sqrt{2}, 0).$$

The other possibility that $x^2 = y^2$ gives the points

$$(x, y, z) = (\pm 2, \pm 2, \pm 4)$$

(with all 8 possibilities of $\pm$ for the three coordinates). These are the only points that satisfy the equation

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

so the maximum and minimum of $\sqrt{f}$ must be attained at one of these points. Plugging in these points, we find that $\sqrt{f}$ attains its minimum value of $2\sqrt{2}$ at

$$(x, y, z) = \left(\pm 2\sqrt{2}, 0, 0\right) \quad \text{and} \quad (0, \pm 2\sqrt{2}, 0)$$

and its maximum value of $4\sqrt{2}$ at

$$(x, y, z) = (\pm 2, \pm 2, \pm 8).$$

$\square$