A NASTY PARTIAL-FRACTIONS/TRIG INTEGRAL

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Exercise 44 of §8.5 in your book asks to compute the following integral:

$$\int \frac{x^2+3}{(x^2+2x+3)^2}\,dx.$$

To do this, we first want to notice that the integrand is a *proper* rational function; that is, the degree of the denominator (4, in this case), is greater than the degree of the numerator (2, in this case). Also notice that the polynomial x^2+2x+3 is an irreducible quadratic, since its discriminant

$$b^2-4ac = 2^2-4(1)(3) = 4-12$$

is negative. So partial fractions should work here: there are constants A, B, C, and D such that

$$\frac{x^2+3}{(x^2+2x+3)^2} = \frac{Ax+B}{x^2+2x+3} + \frac{Cx+D}{(x^2+2x+3)^2}. \tag{1}$$

Multiplying both sides by $(x^2+2x+3)^2$ to clear denominators and then distributing gives

$$\begin{aligned} x^2+3 &= (Ax+B)(x^2+2x+3)+Cx+D \\ &= Ax(x^2+2x+3)+B(x^2+2x+3)+Cx+D \\ &= Ax^3+2Ax^2+3Ax+Bx^2+2Bx+3B+Cx+D \\ &= Ax^3+(2A+B)x^2+(3A+2B+C)x+3B+D. \end{aligned}$$

We have an equation with a polynomial on each side. The only way this can happen is that corresponding coefficients are equal; that is, the coefficient of x^2 on the left should equal the coefficient of x^2 on the right, and the coefficient of x on the left should equal the coefficient of x on the right, etc. We apply this fact to get the following equations:

$$0 = A \tag{2}$$

$$1 = 2A+B \tag{3}$$

$$0 = 3A+2B+C \tag{4}$$

$$3 = 3B+D. \tag{5}$$

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Apply equation (2) to eliminate all of the A terms, so $1 = B$ and $0 = 2B + C = 2 + C$. That is, $-2 = C$. Plugging 1 in for B in equation (5) gives $D = 0$.

Now we go back to equation (1) and plug in the values for A, B, C, and D:

$$\begin{aligned}\frac{x^2+3}{(x^2+2x+3)^2} &= \frac{0x+1}{x^2+2x+3} + \frac{-2x+0}{(x^2+2x+3)^2} \\ &= \frac{1}{x^2+2x+3} - \frac{2x}{(x^2+2x+3)^2}.\end{aligned}$$

Our original integral becomes:

$$\int \frac{x^2+3}{(x^2+2x+3)^2}\,dx = \int \frac{1}{x^2+2x+3} - \frac{2x}{(x^2+2x+3)^2}\,dx$$

Hopefully the fact that x^2+2x+3 has derivative $2x+2$ (which is *almost* $2x$, the numerator of the second fraction) suggests to you that u-substitution might be a good idea for the second term. The problem is that we have only a $2x$, not a $2x+2$. So we add and subtract 2 and split up the fraction as follows:

$$\begin{aligned}\int \frac{x^2+3}{(x^2+2x+3)^2}\,dx &= \int \frac{1}{x^2+2x+3} - \frac{2x}{(x^2+2x+3)^2}\,dx \\ &= \int \frac{1}{x^2+2x+3} - \frac{2x+2-2}{(x^2+2x+3)^2}\,dx \\ &= \int \frac{1}{x^2+2x+3} - \frac{2x+2}{(x^2+2x+3)^2} + \frac{2}{(x^2+2x+3)^2}\,dx \\ &= \int \frac{1}{x^2+2x+3} + \frac{2}{(x^2+2x+3)^2}\,dx - \int \frac{2x+2}{(x^2+2x+3)^2}\,dx. \qquad (6)\end{aligned}$$

The second integral should be no problem, since we arranged for it to be a straightforward u-substitution. Letting $u = x^2+2x+3$, we get $du = (2x+2)dx$ and

$$-\int \frac{2x+2}{(x^2+2x+3)^2}\,dx = -\int \frac{du}{u^2} = \frac{1}{u} + C = \frac{1}{x^2+2x+3} + C. \qquad (7)$$

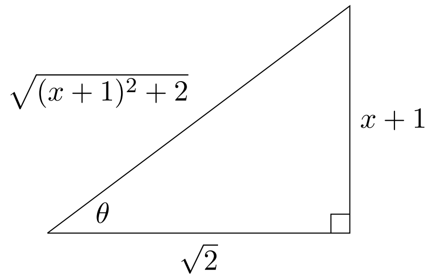
Unfortunately the first integral isn't so easy. We'll need to complete the square and use trig substitution. Notice that $x^2+2x+1 = (x+1)^2$, so $x^2+2x+3 = (x+1)^2+2$. Use this fact to rewrite the integral we're trying to evaluate:

$$\int \frac{1}{x^2+2x+3} + \frac{2}{(x^2+2x+3)^2}\,dx = \int \frac{1}{(x+1)^2+2} + \frac{2}{((x+1)^2+2)^2}\,dx.$$

The expression $(x+1)^2+2$ is of the form u^2+a^2, so we should think of using trig substitution with a substitution

$$x+1 = \sqrt{2}\tan\theta. \qquad (8)$$

(You can think of this as a u-substitution $u = x+1$ followed by the trig substitution $u = \sqrt{2}\tan\theta$, but this isn't necessary.) Now draw a picture:



We need to express $\frac{1}{(x+1)^2+2}$ and $\frac{2}{((x+1)^2+2)^2}$ in terms of θ. Notice first that (look at the picture!)

$$\cos\theta = \frac{\sqrt{2}}{\sqrt{(x+1)^2+2}}, \tag{9}$$

so

$$\frac{1}{(x+1)^2+2} = \frac{1}{2}\cos^2\theta \quad \text{and} \quad \frac{2}{((x+1)^2+2)^2} = \frac{1}{2}\cos^4\theta.$$

Take the derivative of each side of equation (8) to get

$$dx = \sqrt{2}\sec^2\theta\, d\theta.$$

Now we can carry out the substitution:

$$\begin{aligned}
\int \frac{1}{(x+1)^2+2} + \frac{2}{((x+1)^2+2)^2}\, dx &= \int \left(\frac{1}{2}\cos^2\theta + \frac{1}{2}\cos^4\theta\right)\sqrt{2}\sec^2\theta\, d\theta \\
&= \frac{\sqrt{2}}{2}\int (\cos^2\theta + \cos^4\theta)\sec^2\theta\, d\theta \\
&= \frac{\sqrt{2}}{2}\int \left(1 + \frac{\cos^4\theta}{\cos^2\theta}\right) d\theta \\
&= \frac{\sqrt{2}}{2}\int (1+\cos^2\theta)\, d\theta.
\end{aligned}$$

Finish the computation by using the trig identities $\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$ and $\sin(2\theta) = 2\sin\theta\cos\theta$:

$$\begin{aligned}
\frac{\sqrt{2}}{2}\int (1+\cos^2\theta)\, d\theta &= \frac{\sqrt{2}}{2}\left(\int 1\, d\theta + \int \cos^2\theta\, d\theta\right) \\
&= \frac{\sqrt{2}}{2}\left(\theta + \frac{1}{2}\int (1+\cos(2\theta))\right) d\theta \\
&= \frac{\sqrt{2}}{2}\left(\theta + \frac{1}{2}\theta + \frac{\sin(2\theta)}{4}\right) + C \\
&= \frac{3\sqrt{2}}{4}\theta + \frac{\sqrt{2}}{8}\sin(2\theta) + C \\
&= \frac{3\sqrt{2}}{4}\theta + \frac{\sqrt{2}}{4}\sin\theta\cos\theta + C \qquad (10)
\end{aligned}$$

Now we need to plug x-things back in for the θ-things. Go back and solve equation (8) for θ to get

$$\theta = \arctan\left(\frac{x+1}{\sqrt{2}}\right)$$

Recall (look at the picture!) that

$$\sin\theta = \frac{x+1}{\sqrt{(x+1)^2+2}}.$$

Multiply this by the right side of equation (9) to get

$$\sin\theta\cos\theta = \left(\frac{x+1}{\sqrt{(x+1)^2+2}}\right)\left(\frac{\sqrt{2}}{\sqrt{(x+1)^2+2}}\right) = \frac{\sqrt{2}(x+1)}{(x+1)^2+2}.$$

This is the last ingredient we need to replace the θ-things in equation (10) with x-things:

$$\begin{aligned}\frac{\sqrt{2}}{2}\int(1+\cos^2\theta)\,d\theta &= \frac{3\sqrt{2}}{4}\theta + \frac{\sqrt{2}}{4}\sin\theta\cos\theta + C \\ &= \frac{3\sqrt{2}}{4}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + \frac{\sqrt{2}}{4}\cdot\frac{\sqrt{2}(x+1)}{(x+1)^2+2} + C \\ &= \frac{3\sqrt{2}}{4}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + \frac{1}{2}\cdot\frac{x+1}{(x+1)^2+2} + C.\end{aligned}$$

In summary,

$$\begin{aligned}\int \frac{1}{(x+1)^2+2} + \frac{2}{((x+1)^2+2)^2}\,dx &= \frac{\sqrt{2}}{2}\int(1+\cos^2\theta)\,d\theta \\ &= \frac{3\sqrt{2}}{4}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + \frac{1}{2}\cdot\frac{x+1}{(x+1)^2+2} + C. \qquad (11)\end{aligned}$$

Combining our answer in equation (7) with our answer in (11) gives the final answer:

$$\begin{aligned}\int \frac{x^2+3}{(x^2+2x+3)^2}\,dx &= \int \frac{1}{x^2+2x+3} - \frac{2x}{(x^2+2x+3)^2}\,dx \\ &= \int \frac{1}{(x+1)^2+2} + \frac{2}{((x+1)^2+2)^2}\,dx - \int \frac{2x+2}{(x^2+2x+3)^2}\,dx \\ &= \frac{3\sqrt{2}}{4}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + \frac{1}{2}\cdot\frac{x+1}{(x+1)^2+2} + \frac{1}{x^2+2x+3} + C. \\ &= \frac{3\sqrt{2}}{4}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + \frac{1}{2}\cdot\frac{x+1}{x^2+2x+3} + \frac{1}{x^2+2x+3} + C. \\ &= \boxed{\frac{3\sqrt{2}}{4}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + \frac{1}{2}\cdot\frac{x+3}{x^2+2x+3} + C.}\end{aligned}$$