MATH 131A: SOME QUIZ SOLUTIONS

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Problem 3. Let L be a real number. Show that a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ converges to L if and only if every subsequence $(a_{k_n})_{n\in\mathbb{N}}$ has itself a subsequence converging to L.

Solution. (only if) A theorem (page ??) in your textbook says that a sequence converges to L iff every one of its subsequences converges to L. So if $(a_n)_{n\in\mathbb{N}}$ converges to L and $(a_{k_n})_{n\in\mathbb{N}}$ is a subsequence, then $(a_{k_n})_{n\in\mathbb{N}}$ has a subsequence (itself!) that converges to L.

(if) We prove the contrapositive. Suppose that $a_n \not\to L$. We must produce a subsequence $(a_{k_n})_{n\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ such that none of its subsequences converge to L. Consider what it means (by definition) for $a_n \not\to L$: there is a "bad" $\epsilon > 0$ such that

$$(\star) \qquad (\forall N \in \mathbb{N})(\exists m > N)\ |a_m - L| \geq \epsilon.$$

Now just enumerate the terms of the sequence that are $\geq \epsilon$ away from L. More precisely, define the subsequence $(a_{k_n})_{n\in\mathbb{N}}$ as follows. Let k_0 be least such that $|a_{k_0} - L| \geq \epsilon$. Inductively assume that $k_0, \dots, k_n$ are defined such that

(1) $k_0 < k_1 < \cdots < k_n$, and
(2) $|a_{k_m} - L| \geq \epsilon$ for all $m = 0, 1, \dots, n$.

Apply $(\star)$ with $N = k_n$ to get $m > k_n$ such that $|a_{k_m} - L| \geq \epsilon$. Define $k_{n+1} = m$. This satisfies conditions (1) & (2) above, so the induction is complete. We have a subsequence $(a_{k_n})_{n\in\mathbb{N}}$ such that $|a_{k_n} - L| \geq \epsilon$ for every $n \in \mathbb{N}$. Every term of $(a_{k_n})_{n\in\mathbb{N}}$ is $\geq \epsilon$ away from L, so in particular every term of every subsequence of $(a_{k_n})_{n\in\mathbb{N}}$ is $\geq \epsilon$ away from L. (That is, since $(a_{k_n})_{n\in\mathbb{N}}$ is bounded away from L, every subsequence of $(a_{k_n})_{n\in\mathbb{N}}$ is bounded away from L.) This reduces the problem to the following.

Claim. Let $\epsilon > 0$ and let $(b_n)_{n\in\mathbb{N}}$ be a sequence. Suppose that $|b_n - L| \geq \epsilon$ for every $n \in \mathbb{N}$. Then $(b_n)_{n\in\mathbb{N}}$ does not converge to L.

Proof of claim. We have to show that there is some $\epsilon > 0$ such that $|b_n - L| \geq \epsilon$ for infinitely many $n \in \mathbb{N}$. This is immediate from the assumption of the claim. Indeed, for every $N \in \mathbb{N}$, it's clear that $m = N + 1$ satisfies $m > N$ and $|a_m - L| \geq \epsilon$, since $|a_n - N| \geq \epsilon$ for **every** $n \in \mathbb{N}$ (not just those n that are greater than N). ∎

I'll reiterate why the claim finishes the proof: Apply the claim to any subsequence of $(a_{k_n})_{n\in\mathbb{N}}$ to see that no subsequence of $(a_{k_n})_{n\in\mathbb{N}}$ converges to L. ∎

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Problem 4. Suppose that A is a nonempty subset of $\mathbb{R}$ and that $\sup(A) \notin A$. Show that there is a sequence $(a_n)_{n\in\mathbb{N}}$ in A that is convergent to $\sup(A)$.

Solution. Since the problem is asking us to prove something about $\sup(A)$, it's fair to assume A is bounded above (so that $\sup(A)$ actually exists!). The point is to define the sequence $(a_n)_{n\in\mathbb{N}}$, which we'll do by repeatedly using the definition of supremum for various values of ϵ.

Since $\sup(A) - 1$ is not an upper bound for A (it's less than the least upper bound!), we can choose $a_0 \in A \cap (\sup(A) - 1, \sup(A)]$. Now assume inductively that $a_0, \ldots, a_n$ are defined so that for all $k \in \{0, 1, \ldots, n\}$:

- $a_k \in A$, and
- $\sup(A) - \frac{1}{k+1} < a_k$.

Since $\sup(A) - \frac{1}{n+2}$ is not an upper bound for A, there is $a_{n+1} \in A$ such that $a_{n+1} > \sup(A) - \frac{1}{n+2}$. This completes the inductive construction, giving a sequence $(a_n)_{n\in\mathbb{N}}$ with terms in A such that for every $n \in \mathbb{N}$

$$\sup(A) - \frac{1}{n+1} < a_n \leq \sup(A).$$

(The first inequality comes from our construction, and the second is just from the definition of supremum.) Since $\sup(A) - \frac{1}{n+1} \to \sup(A)$, we can apply the sandwich theorem to conclude that $a_n \to \sup(A)$, as required. ∎

Remarks.

- The hypothesis $\sup(A) \notin A$ is unnecessary, and this proof doesn't use it.
- With a little extra care, we could have ensured that the sequence $(a_n)_{n\in\mathbb{N}}$ was increasing, which (together with the fact that $(a_n)_{n\in\mathbb{N}}$ is bounded) would guarantee that $(a_n)_{n\in\mathbb{N}}$ converges. This is fine, but it isn't necessary. It's important to realize that our proof as it is doesn't necessarily give an increasing sequence.