HOW TO PROVE THAT A NON-REPRESENTABLE FUNCTOR IS NOT REPRESENTABLE

ZACH NORWOOD

Health Warning: I wrote these notes for myself. They come with no guarantee that they'll be helpful to you. In fact, it's likely that some of the commentary in these notes will make a nonpositive contribution to your understanding of this material. That aside, if you find anything in these notes that's definitely false, please let me know.

The purpose of this short set of notes is to discuss some old qual problems related to representability of functors $\mathcal{C} \to \mathbf{Sets}$. In particular, we will show how to prove that a non-representable functor is not representable.

Recall that a functor $F \colon \mathcal{C} \to \mathbf{Sets}$ is **representable** if there is an object (a **representing object**) $A \in \operatorname{ob} \mathcal{C}$ such that F is naturally isomorphic to the hom-functor $\hom(A, -)$. As we will see, many natural examples of representable functors are contravariant. (Sometimes these are called **corepresentable**, but we won't use that term.) A contravariant functor $F \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Sets}$ is representable if it is naturally isomorphic to $\hom(-, A)$ for some $A \in \operatorname{ob} \mathcal{C}$.

Note: I will probably write $\mathcal{C}(A, -)$ for $\hom(A, -)$ occasionally. Sorry if this upsets you.

It would be a crime to say anything about representable functors without mentioning the Yoneda Lemma:

Theorem 1 (Yoneda Lemma)**.** Let $\mathcal{C}$ be a locally small category, $A \in \operatorname{ob} \mathcal{C}$, and $F \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Sets}$ a functor. There is a bijection

$$\hom(\mathcal{C}(-, A), F) \to FA$$

natural in both A and F. (Here the hom-set is the hom-set from the functor category $[\mathcal{C}^{\mathrm{op}}, \mathbf{Sets}]$, i.e., the collection of natural transformations between its arguments.)

Its most important corollary is:

Corollary 1. The Yoneda embedding

$$A \mapsto \hom(-, A) \colon \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Sets}]$$

is full and faithful.

My favorite application, obtained by applying the Yoneda Lemma to a group, considered as a category with one object:

Corollary 2 (Cayley's Theorem)**.** Every group embeds into a symmetric group.

Date: 27 Feb 2013.

Here is a long list of examples.

Examples. You should prove any assertion here that you don't believe. In particular, you should convince yourself that each of these gadgets is actually the action-on-objects of a functor.

(1) The forgetful functor **Grps** $\to$ **Sets** is represented by the group $\mathbb{Z}$.

(2) The forgetful functor **Rings** $\to$ **Sets** is represented by the ring $\mathbb{Z}[X]$.

(3) The forgetful functor **Top** $\to$ **Sets** is represented by the one-point space.

(4) The functor $(-)^\times\colon$ **Rings** $\to$ **Sets** that sends a ring to its set of units is representable. (Exercise (old qual problem): Find a representing object!)

(5) The contravariant powerset functor $\mathscr{P}\colon$ **Sets** $\to$ **Sets** is representable.

(6) The (contravariant) functor **Top** $\to$ **Sets**, $X \mapsto \mathcal{O}(X)$, that sends a topological space to its set of open sets is represented by the two-point space equipped with the only (up to renaming the points) nontrivial non-Hausdorff topology on it.

Now on to the main event. After being handed a non-representable functor, how do we prove that it's not representable? The idea is to assume that the functor is representable, transfer the problem to a concrete setting by considering a 'universal element' for the functor, and then derive a contradiction by proving that the universal element cannot have the outrageous universal property that it's supposed to have. Let's illustrate with an example:

Example (Fall 2012, #4)**.** The functor $F\colon$ **Rings** $\to$ **Sets** that sends a ring A to $\{a^2 : a \in A\}$, its set of squares, is not representable.

(First make sure you believe that this is a (covariant) functor!)

Proof. Suppose for a contradiction that $F \cong \hom(A, -)$, so that in particular $FA \simeq \hom(A, A)$ in **Sets**. Let $u \in FA$ correspond via this isomorphism to id_A. (This u is the universal element.) Then u is a square in A; say $u = a^2$. We will show that u has the following universal property: for every ring B and every square $b^2 \in B$, there is a unique homomorphism $A \to B$ sending u to b^2. (Seems pretty unlikely.) Let α be a natural isomorphism $\hom(A, -) \Rightarrow F$, and consider the naturality square:

$$\begin{array}{ccc}
\hom(A, A) & \xrightarrow{\;g \circ -\;} & \hom(A, B) \\
\wr\downarrow \alpha_A & & \alpha_B \downarrow\wr \\
FA & \xrightarrow{\;Fg\;} & FB
\end{array}$$

The map $\mathrm{id}_A \in \hom(A, A)$ has the universal property that every $g \in \hom(A, B)$ is the unique member of $\hom(A, B)$ such that $(g \circ -)(\mathrm{id}_A) = g$. (This is a triviality.)

Therefore, by naturality, $u \in FA$ has the universal property that for every square $b^2 \in FB$, there is a unique homomorphism $g\colon A \to B$ such that $(Fg)(u) = b^2$, i.e., $g(u) = b^2$. (If you believe this, skip the next paragraph.)

To wit: Let $b^2 \in FB$, and put $g = \alpha_B^{-1}(b^2)$. Then we have

$$(Fg)(u) = (Fg)(\alpha_A(\mathrm{id}_A)) = \alpha_B(g \circ -)(\mathrm{id}_A) = \alpha_B(g) = b^2,$$

since the naturality square commutes. Suppose that g' is another homomorphism $A \to B$ such that $(Fg')(u) = b^2$. But then

$$(g' \circ -)(\mathrm{id}_A) = \alpha_B^{-1}(Fg')(\alpha_A)(\mathrm{id}_A) = \alpha_B^{-1}(Fg)(u) = \alpha_B^{-1}(b^2) = g,$$

which implies that $g = g'$ by the universal property of $\mathrm{id}_A \in \hom(A, A)$. ✓

We have shown that $u = a^2$ has the universal property that for every square b^2 in any ring B, there is a unique homomorphism $A \to B$ sending u to b^2. Now we have transferred the problem to ring theory, where we will finish it off.

Let $B = \mathbb{Z}[X]$, $b = X$. There is a (supposedly unique) homomorphism $g\colon A \to \mathbb{Z}[X]$ such that $g(u) = X^2$. This means that $g(a) \in \{\pm X\}$. Let $\phi\colon \mathbb{Z}[X] \to \mathbb{Z}[X]$ be the unique automorphism interchanging X and $-X$. Now g and $\phi \circ g$ are two distinct homomorphisms $A \to \mathbb{Z}[X]$ sending u to X^2. This is the required contradiction. ∎

Notice that $X^2 \in \mathbb{Z}[X]$ is *almost* a universal element of the sort we showed can't exist, in the sense that there is always at least one homomorphism sending X^2 to any given square in any given ring. The problem is that there can be more than one homomorphism.

Examples. Use this technique to prove the following facts.

(a) Consider the function from commutative rings to sets, sending a ring to its set of nilpotent elements. This functor is not representable (though for a fixed n the functor $A \mapsto \{a \in A : a^n = 0\}$, is represented by $\mathbb{Z}[X]/(X^n)$).

(b) (Spring 2011 #10) Show that the (covariant) functor **Grps** $\to$ **Sets** sending a group to its set of subgroups is not representable.

(c) Prove that the functor from the category of Hausdorff spaces to **Sets** sending a space to its set of open sets is not representable. (Compare to Example (6) above.)

(d) The covariant powerset functor $\mathscr{P}\colon$ **Sets** $\to$ **Sets** is not representable. (Compare to Example (5) above.)