TYCHONOFF'S THEOREM IMPLIES AC

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For our purposes, the *Axiom of Choice* (AC) is the following statement:

> A product of nonempty sets is nonempty. That is, if $\{S_\alpha : \alpha \in A\}$ is a set of nonempty sets, then the product $\prod_{\alpha \in A} S_\alpha$ is nonempty.[1]

Recall that Tychonoff's Theorem is the assertion that a product of compact topological spaces is compact. We will show (without using AC) that Tychonoff's Theorem implies AC. It follows that Tychonoff's Theorem is equivalent to AC, since there is a proof (using AC) of Tychonoff's Theorem. This equivalence was first observed by Kelley in 1950 ([1]).

It will turn out to be useful (as it often is in applications of compactness) to use the following form of the definition of compactness, which is equivalent to the usual one.

Definition. A family $\mathfrak{F}$ of sets has the *finite intersection property* (FIP) if, for all $F_1, \ldots, F_n \in \mathfrak{F}$, the intersection $F_1 \cap \cdots \cap F_n$ is nonempty.

A space X is compact iff it has the following property: if $\mathfrak{F}$ is a family of closed subsets of X with the FIP, then $\bigcap \mathfrak{F} \neq \varnothing$; that is, there is a point $x \in X$ that belongs to every $F \in \mathfrak{F}$. You should write down the three-line proof that this condition is equivalent to compactness. The proof requires little more than the fact that a set is closed iff its complement is open.

Theorem. Tychonoff's Theorem implies AC.

Proof. Let $\{S_\alpha : \alpha \in A\}$ be a set of nonempty sets. Our task is to show that the product $\prod_{\alpha \in A} S_\alpha$ is nonempty. Add to each set S_α a new element d that doesn't belong to any of the S_αs. Now topologize $S_\alpha \cup \{d\}$ as follows: equip S_α with the cofinite topology (the open sets are the cofinite sets and the empty set). And declare that the set $\{d\}$ is open; that is, d should be an isolated point in $S_\alpha \cup \{d\}$.

Any space with the cofinite topology is compact. The idea: to produce a finite subcover of a typical open cover, take one nonempty set from the open cover. It is cofinite, so there are only finitely many points that don't belong

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[1]This is what I call the 'Algebraist's form of the Axiom of Choice'. The point is that there is a *function* that assigns to each α a member of S_α. I think the algebraist's version makes that less clear than it should be.

to it. Pick one set from the cover for each of those remaining points, and you have a finite subcover. (Write this down carefully!)

Adding an isolated point to a compact space preserves compactness (meditate on this for a minute and you'll see why it's true), so $S_\alpha \cup \{d\}$ is compact. By Tychonoff's theorem, the product space $X = \prod_{\alpha \in A}(S_\alpha \cup \{d\})$ is compact. We must use this to conclude that the set $\prod_{\alpha \in A} S_\alpha$ is nonempty. Consider the following family of subsets of the product space:

$$\mathfrak{F} = \left\{ \pi_\alpha^{-1}(S_\alpha) : \alpha \in A \right\},$$

where $\pi_\alpha \colon X \twoheadrightarrow S_\alpha \cup \{d\}$ is the map that projects onto coordinate α. Since S_α is closed in $S_\alpha \cup \{d\}$ (its complement is open!), and the projection π_α is continuous, every member of the family $\mathfrak{F}$ is a closed subset of X. We will verify that $\mathfrak{F}$ has the FIP. Let $F_1, \dots, F_n \in \mathfrak{F}$; say

$$F_1 = \pi_{\alpha_1}^{-1}(S_{\alpha_1}), \dots, F_n = \pi_{\alpha_n}^{-1}(S_{\alpha_n}).$$

This is a subtle step, but the point is that we don't need AC to prove that a *finite* product of sets is nonempty. (Prove it by induction!) Since each S_{α_k} is nonempty, there is some $s_k \in S_{\alpha_k}$, and we can use these, together with the distinguished point d, to produce a member x of $F_1 \cap \cdots \cap F_n$:

$$x(\beta) = \begin{cases} s_k & \text{if } \beta = \alpha_k \text{ for some } k \\ d & \text{otherwise.} \end{cases}$$

Then $x \in F_1 \cap \cdots \cap F_n$, so $\mathfrak{F}$ has the FIP.

By the compactness of X, there is some $x \in X$ that belongs to every member of $\mathfrak{F}$; that is, $x \in \bigcap_{\alpha \in A} \pi_\alpha^{-1}(S_\alpha)$. But $\bigcap_{\alpha \in A} \pi_\alpha^{-1}(S_\alpha)$ is just the set of y whose α^{th} entry belongs to S_α, for every α, and this set is just $\prod_{\alpha \in A} S_\alpha$. So we have shown that $\prod_{\alpha \in A} S_\alpha$ is nonempty, as required. ∎

References

[1] Kelley, J. L. *The Tychonoff product theorem implies the axiom of choice.* Fund. Math. 37, (1950). 75–76. (MR 39982)