FILTER CONVERGENCE AND TYCHONOFF'S THEOREM

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Definition. A **filter** on a set S is a nonempty family F of subsets of S such that (i)–(iv) hold:

(i) $\varnothing \notin F$;

(ii) if $A \in F$ and $B \supseteq A$, then $B \in F$;

(iii) if $A, B \in F$ then $A \cap B \in F$ (so by induction F is closed under all finite intersections);

(iv) $S \in F$ (this follows from (i) and (ii), but it's worth mentioning anyway).

If F also satisfies

(v) for all $A \subseteq S$, either $A \in F$ or $S \setminus A \in F$ (but not both, by (iii) and (iv)),

then F is an **ultrafilter**.

A filter defines a notion of largeness for subsets of S: the large sets are the members of S, and the small sets are the sets whose complements are large. An ultrafilter demands that every set be either large or small. It is not difficult to show that a filter F is maximal (meaning no filter properly contains it) iff F is an ultrafilter.

Perhaps the most important examples of filters are the following:

(1) If a is any member of S, then the family $\{A \subseteq S : a \in A\}$ is a filter on S. In fact it is an ultrafilter, since every subset A of S satisfies either $a \in A$ or $a \notin A$. When an ultrafilter is generated by a single element like this, we say the ultrafilter is **principal**.

(2) More generally, let A be a nonempty subset of S, and let F be the family of all supersets of A. Then F is a filter, but an ultrafilter only if A is a singleton $\{a\}$.

(3) Let S be infinite, and let F be the family of all cofinite subsets of S. The empty set is not cofinite (since S is infinite), a superset of a cofinite set is still cofinite, the intersection of two cofinite sets is cofinite, and $S \in F$, so F satisfies conditions (i)–(iv) and is therefore a filter. We call F the **Fréchet** filter or the **cofinite** filter. The cofinite filter is never an ultrafilter: if $S = \mathbb{N}$, for instance, then the set of even numbers is

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neither finite nor cofinite, so neither it nor its complement is a member of F.

(4) The family of neighborhoods of a point x in a space X is a filter. Recall that N is a neighborhood of x if N contains an open set U such that $x \in U$, i.e., $x \in U \subseteq N$.

We have so far seen only one example of an ultrafilter. Are there more? An ultrafilter that is not principal is called **nonprincipal** or **free**, and such an ultrafilter must contain the cofinite filter. (More generally, a filter on S is **free** if there is no element of S that belongs to every set in the filter.) To see this, suppose that F is a nonprincipal ultrafilter, so that $\{a\}^c \in F$ for every $a \in S$. Suppose that some finite set $A \subseteq S$ belongs to F. Since F is closed under finite intersection, it follows that

$$A \cap \bigcap_{a \in A} \{a\}^c = \varnothing \in F,$$

a contradiction. So no finite set belongs to F. But F is an ultrafilter, so the complements of all of the finite sets must belong to F. That is, F contains the cofinite filter. But are there any such ultrafilters? Yes, but their existence depends on the Axiom of Choice.

Theorem (Ultrafilter Lemma)**.** Every filter is contained in an ultrafilter. That is, if F is a filter on S, then there is an ultrafilter G on S such that $F \subseteq G$.

One proves the Ultrafilter Lemma by a standard Zorn's Lemma argument. (Think of the proof that every linearly independent subset of a vector space is contained in a basis, i.e., a maximal linearly independent set.) The Ultrafilter Lemma is not provable without some form of the Axiom of Choice, though it is strictly weaker than AC. In fact, the Ultrafilter Lemma is equivalent to Tychonoff's theorem for Hausdorff spaces, while AC is equivalent to Tychonoff's theorem for all spaces.

On that note, a quick reminder: for our purposes, the **Axiom of Choice** (AC) is the following statement:

> A product of nonempty sets is nonempty. That is, if $\{S_\alpha : \alpha \in A\}$ is a set of nonempty sets, then the product $\prod_{\alpha \in A} S_\alpha$ is nonempty.

If a family of sets contains, say, two disjoint sets, then it can't possibly extend to a filter, since the filter would have to contain the intersection of those two sets, i.e., $\varnothing$. What conditions on a family guarantee that the family extends to a filter? We need the family to satisfy the FIP.

Definition. Let F be a family of subsets of S. We say that F has the **finite intersection property** (FIP) if all finite subfamilies of F have nonempty intersection; that is, if for all $A_1, \ldots, A_n \in F$,

$$A_1 \cap \cdots \cap A_n \neq \varnothing.$$

A family F with the FIP extends to a filter in the following way. Take all finite intersections of members of F, and then take all supersets of those. This is a filter, as you should check.

In our analysis of filters on topological spaces, we will need to consider the **pushforward** of a filter F along a function $f \colon S \to T$. We define the pushforward $f_* F$ to be the family $\{B \subseteq T : f^{-1}[B] \in F\}$.

The proof that the pushforward (along $f \colon X \to Y$) of a filter (on X) is a filter (on Y) is routine and short. We should also verify that if F is an ultrafilter, then the pushforward $f_* F$ is too. Let $B \subseteq Y$. Since F is an ultrafilter, either $f^{-1}[B] \in F$ or $(f^{-1}[B])^c \in F$. Notice that $(f^{-1}[B])^c = f^{-1}[B^c]$, so either $f^{-1}[B] \in F$ or $f^{-1}[B^c] \in F$. This says exactly that $B \in f_* F$ or $B^c \in f_* F$, so $f_* F$ is an ultrafilter.

The first hint of a connection between filters and compactness is the following version of compactness. A space X is compact iff it has the following property: if F is a family of subsets of X with the FIP, then $\bigcap_{A \in F} \overline{A} \neq \varnothing$; that is, there is a point $x \in X$ that belongs to the closure of every member of F.

Now we are ready to connect filters to topology.

Definition. Let F be a filter on X, a topological space. We say that F **converges** to the point $x \in X$, and we write $F \to x$, if every open neighborhood of x is a member of F.

Intuitively, every open neighborhood of x is large, according to the filter. Notice that, as is the case with sequences, we have no reason to think that the limit of a filter is unique, if it has one. As a special case of this notion of convergence, we recover the usual convergence of sequences. A sequence is just a function $f \colon \mathbb{N} \to X$, and it converges to a iff every open neighborhood of a contains cofinitely many terms of the sequence. This is exactly the assertion that the pushforward of the cofinite filter along the function f converges to a.

Where sequences fail to describe the topology[1], ultrafilters succeed. It would be unfair to mention generalized convergence in topological spaces without also mentioning nets, which rival filters in providing a satisfactory theory of convergence in general spaces.

Filter convergence was originally formulated by Henri Cartan around 1937 and explored by Bourbaki in the 1940s.

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[1] Sequential continuity doesn't imply continuity, the closure of a set isn't always just the set of limits of sequences with terms from the set, sequential compactness is not compactness, etc.

Exercise. Let X be a topological space.

(1) Let B be a nonempty subset of X. Then $x \in \overline{B}$ iff there is a filter F on X such that $B \in F$ and $F \to x$.

(2) A function $f\colon X \to Y$ is continuous iff the following condition holds: for every filter F on X, if $F \to x$, then $f_*F \to f(x)$.

The results of this exercise are just what we hoped would work — but didn't — for sequences in general spaces. Remarkably, we also have a characterization of compactness in terms of filters.

Theorem. A topological space X is compact iff every ultrafilter on X converges to at least one point.

Proof. Suppose first that X is compact, and let F be an ultrafilter on X. Then F has the FIP, since it is closed under finite intersections, and $\varnothing \notin F$. Compactness then guarantees that there is some point $x \in \bigcap_{B \in F} \overline{B}$. This means that every open neighborhood of x meets every $B \in F$. Let U be an open neighborhood of x. Since no member of F is disjoint from U, we see that in particular $U^c \notin F$. Since F is an ultrafilter, it must be that $U \in F$. This proves that F converges to x.

For the converse, suppose that every ultrafilter converges and let F be a family of subsets of X that has the FIP. Then F generates a filter, which can then be extended to an ultrafilter G. By assumption, G converges to some point x. Consider $B \in F$. Since $G \to x$, every neighborhood of x meets B. This says exactly that $x \in \overline{B}$, so, since this is true of every $B \in F$, we have $x \in \bigcap_{B \in F} \overline{B}$. This proves that X is compact. ∎

There is a similar nice characterization of Hausdorff spaces in terms of (ultra)filters. I leave it as an exercise, since it won't be necessary for our proof of Tychonoff's theorem.

Exercise. A topological space X is Hausdorff iff every ultrafilter on X converges to at most one point.

So in a compact Hausdorff space, every ultrafilter has a unique limit.

Now we can prove Tychonoff's Theorem.

Theorem (Tychonoff)**.** A product of compact spaces is compact.

Proof. Let X_α, $\alpha \in A$, be compact topological spaces, and set $X = \prod_{\alpha \in A} X_\alpha$, with projection maps $\pi_\alpha\colon X \twoheadrightarrow X_\alpha$. We need to show that every ultrafilter F on X converges to at least one point. The heart of the matter is the following observation:

Claim. If F is a filter on X, then $F \to x$ in X if and only if $(\pi_\alpha)_*F \to x_\alpha$ for every $\alpha \in A$.

Suppose first that $F \to x$, and fix $\alpha \in A$. (This implication follows from an exercise above and the fact that the projections π_α are continuous, but let's just prove it anyway.) If U is an open neighborhood of x_α in X_α, then $\pi_\alpha^{-1}[U]$ is an open neighborhood of x, and it belongs to F by assumption. Therefore U belongs to the pushforward $(\pi_\alpha)_*F$, by definition of 'pushforward'. So $(\pi_\alpha)_*F \to x_\alpha$.

For the converse, suppose that $(\pi_\alpha)_*F \to x_\alpha$ for every $\alpha \in A$, and let $U \subseteq X$ be a basic open neighborhood of x. There is a basic open neighborhood B of x such that

$$B = \pi_{\alpha_1}^{-1}[V_{\alpha_1}] \cap \cdots \cap \pi_{\alpha_n}^{-1}[V_{\alpha_n}] \subseteq U.$$

We need to prove that $U \in F$, and it suffices to prove that $B \in F$, since filters are closed under taking supersets. Observe that V_{α_k} is an open neighborhood of x_{α_k} for all $k \in \{1, \ldots, n\}$, so $V_{\alpha_k} \in (\pi_{\alpha_k})_*F$ by our assumption that $(\pi_\alpha)_*F \to x_\alpha$ for every α. This means that $\pi_{\alpha_k}^{-1}[V_{\alpha_k}] \in F$ for all $k \in \{1, \ldots, n\}$, and it follows that $B \in F$, since F is closed under finite intersections. So $U \in F$, and $F \to x$. This completes the proof of the claim. ✓

To prove the theorem, we need to show that every ultrafilter F on X converges to at least one point. For each α, the pushforward $(\pi_\alpha)_*F$ converges to some points in X_α, since X_α is compact. Let L_α be the set of points in X_α to which $(\pi_\alpha)_*F$ converges. Then L_α is nonempty for each $\alpha \in A$, so by the Axiom of Choice the product $\prod_{\alpha\in A} L_\alpha$ is nonempty. And the claim guarantees that F converges to any member of $\prod_{\alpha\in A} L_\alpha$. We have proved that every ultrafilter on X converges to at least one point, so X is compact. ∎

If we assume that each X_α is compact Hausdorff, then an ultrafilter in X_α will converge to exactly one point. That is, each L_α will have exactly one member, so we don't need to appeal to AC to conclude that $\prod_{\alpha\in A} L_\alpha$ is nonempty.[2] But the equivalence between compactness and convergence of ultrafilters depends on the ultrafilter lemma, as you'll see if you review our proof. So our proof specializes to a proof of Tychonoff's theorem for Hausdorff spaces from the ultrafilter lemma. (The ultrafilter lemma is strictly weaker than AC. You'll have to take my word for it.)

[2]This is worth meditating on, if you haven't thought much about AC before. When there is a 'canonical' choice given for each X_α, then there is a choice function that makes the canonical choice for each α, so no appeal to AC is necessary.