# LOGIC EXERCISES – DAY 1

**Exercise 1.**

(a) If $h\colon \mathbf{A} \to \mathbf{B}$ is an $\mathcal{L}$-homomorphism then its range $h(A)$ is the underlying set of a substructure of $\mathbf{B}$.
(b) Give an example of a homomorphism that isn't a strong homomorphism.
(c) Give an example a strong homomorphism that isn't an embedding.

**Exercise 2.**

(a) Suppose that $X$ is a finite set and $\mathcal{L}$ is a finite language. Prove that there are only finitely many $\mathcal{L}$-structures with underlying set $X$.
(b) Let $\mathcal{L} = \{c_n : n \in \mathbb{N}\}$ be a language with infinitely many constant symbols, and let $A = \{0, 1\}$. How many $\mathcal{L}$-structures have underlying set $A$? Which of them are isomorphic?
(c) Let $\mathcal{L} = \{P\}$ consist of one unary predicate symbol. Describe all countably infinite $\mathcal{L}$-structures up to isomorphism.
(d) Describe up to isomorphism all structures in the empty language.

**Exercise 3.**

(a) Give an example to show that in the language $\mathcal{L}_{\text{group}} = \{1, \cdot\}$, a substructure of a group need not be a subgroup.
(b) Define a language for groups (different from the one in the first part) so that a substructure of a group is a subgroup.

**Exercise 4.**

(a) Prove that the structures $(\mathbb{N}, 0, S, +, \cdot)$ and $(\mathbb{Q}, 0, 1, +, \cdot)$ have no automorphisms other than the identity maps. (An *automorphism* of an $\mathcal{L}$-structure $\mathbf{A}$ is an isomorphism $\mathbf{A} \to \mathbf{A}$.)
(b) Prove that the structure $(\mathbb{R}, 0, 1, +, \cdot)$ has no automorphisms other than the identity.
(c) (If you know a little more set theory than you might have seen in the prep material) Prove that the universe of sets has no $\in$-automorphisms other than the identity.

**Exercise 5.** Recall from algebra that a bijective homomorphism of groups or rings is an isomorphism (of groups or rings). Prove that if $\mathcal{L}$ contains no relation symbols (is an "algebraic language"), then a bijective $\mathcal{L}$-homomorphism is an isomorphism. But use the language of graphs or of orderings to prove that this can fail when $\mathcal{L}$ does contain relation symbols.

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**Exercise 6.**

(a) Prove that $\langle\emptyset\rangle$, i.e. the substructure of $(\mathbb{R}, 0, 1, +, \cdot)$ generated by $\emptyset$, is the substructure $(\mathbb{N}, 0, 1, +, \cdot)$. Can you describe $\langle\emptyset\rangle$ in general?

(b) It's customary in model theory (though we haven't done this) to forbid empty underlying sets. Give an example to show that $\langle\emptyset\rangle$ needn't exist if structures are required to have nonempty underlying sets. (That is, give an example of a structure in which there is no smallest nonempty substructure.)

(c) Suppose that $\mathcal{L}$ has only relation symbols and that $\mathbf{A}$ is an $\mathcal{L}$-structure. What is the substructure generated by a set $X \subseteq A$?

**Exercise 7.** A structure $\mathbf{A}$ is called *ultrahomogeneous* if every isomorphism between finitely generated substructures of $\mathbf{A}$ extends to an automorphism of $\mathbf{A}$. Show that $(\mathbb{Q}, <)$ is ultrahomogeneous. Does your proof also work for $(\mathbb{R}, <)$?