LOGIC EXERCISES – DAY 10

Exercise 1. For the purposes of this problem, you are allowed to assume Gödel's Second Incompleteness theorem, i.e. $\mathsf{PA} \nvdash \mathrm{Con}(\mathsf{PA})$, where $\mathrm{Con}(\mathsf{PA})$ is a sentence in the language of arithmetic encoding "PA is consistent" in the natural way. ($\mathrm{Con}(\mathsf{PA}) \equiv \neg\mathbf{Provable}_{\mathsf{PA}}(\ulcorner \bot \urcorner)$.)

(a) Prove that there is a nonstandard model $\mathfrak{N}$ of PA and an a *nonstandard* element a of the underlying set of $\mathfrak{N}$ which is definable.
(b) Prove that there is a nonstandard model $\mathfrak{N} \models \mathsf{PA}$ without a proper elementary substructure.

Exercise 2. Prove that there is a partial recursive function which has no total recursive extension, i.e., there is $e \in \mathbb{N}$ such that there is no total recursive $f \supseteq \varphi_e$.

Exercise 3. Recall that a set $A \subseteq \mathbb{N}$ is Π^0_2 iff we may write $A(x) \iff \forall y \exists z R(x,y,z)$ for some recursive $R \subseteq \mathbb{N}^3$. A set $B \subseteq \mathbb{N}$ is called Π^0_2-*complete* iff B is Π^0_2 and for all Π^0_2 sets A, $A \leq_1 B$, i.e. there exists a 1-1 total recursive function $f\colon \mathbb{N} \longrightarrow \mathbb{N}$ such that $x \in A \iff f(x) \in B$. Prove that $\mathrm{Tot} = \{e \in \mathbb{N} \colon \varphi_e \text{ is total}\}$ is Π^0_2-complete.

The following two problems make use of *Kleene's Second Recursion Theorem*, which states that if $f(e, \vec{x})$ is a partial recursive function then there exists $e_0 \in \mathbb{N}$ such that $\varphi_{e_0}(\vec{x}) = f(e_0, \vec{x})$ for all $\vec{x}$.

Exercise 4. Suppose that f is a total recursive function. Prove or give a counter-example to each of the following:

(a) There is an e such that $W_{f(e)} = \{e\}$.
(b) There is an e such that $W_e = \{f(e)\}$.

Exercise 5. Let $f(e)$ be a partial recursive function such that for all e,

$$W_e = \emptyset \implies f(e)\downarrow$$

Prove that there is some m such that $W_m = \{m\}$ and $f(m)\downarrow$.

Exercise 6. For $A \subseteq \mathbb{N}^2$ let $A_a = \{b \in \mathbb{N} \colon (a,b) \in A\}$.

(a) Let A be recursively enumerable and suppose $n \in \mathbb{N}$ is such that $|A_a| = n$ for all $a \in \mathbb{N}$. Show that A is recursive.
(b) For *every* pair $n > m$ of natural numbers, give an example of an r.e. set A such that for all a A_a has size n or size m, but A is not recursive.

Date: 3 July 2015.

Exercise 7. Instead of finding monochromatic sets, you might try looking for polychromatic ones. Suppose that $[\mathbb{N}]^2$ is colored (using infinitely many colors) in a way that is k-bounded, meaning that each color is used at most k times. Prove that there is an infinite *fully polychromatic* set X, i.e., a set X such that on pairs of elements of X each color is used at most once.

Exercise 8. This exercise gives a different (possibly more tangible) way to think about the non-axiomatizability of wellfoundedness. (See #5 from Tuesday's set.) For $f, g \colon \mathbb{N} \to \mathbb{N}$ define $f \leq^* g$ if $\{n \in \mathbb{N} : f(n) > g(n)\}$ is finite. That is, $f \leq^* g$ means that $f(n) \leq g(n)$ for cofinitely many n. Say $f <^* g$ if $f(n) < g(n)$ for cofinitely many n. (Note that $f <^* g$ doesn't simply mean that $f \leq^* g$ and $f \neq g$.)

(a) Find a sequence $f_0, f_1, \ldots$ of functions $\mathbb{N} \to \mathbb{N}$ that is $<^*$-decreasing: $f_0 >^* f_1 >^* f_2 >^* \cdots$.

(b) Let $\mathbf{A} = (\mathbb{N}, <)$ be the ordered structure of natural numbers. Let U be a nonprincipal ultrafilter on ω and form the ultrapower $\mathbf{A}^\omega/U$. (Recall that this means the ultraproduct of structures $\mathbf{M}_i$ where $\mathbf{M}_i = \mathbf{A}$ for every i.) Use your answer to part (a) to give an explicit strictly decreasing sequence in the ultrapower $(\mathbf{A}^\omega/U, <^{\mathbf{A}^\omega/U})$. Of course, by Łoś's theorem this ultrapower is elementarily equivalent to $\mathbf{A}$, so deduce again that the class of wellorders is not axiomatizable.