LOGIC EXERCISES – DAY 11

Exercise 1. Prove that the adjective "total" in the definition of "r.e." is unnecessary. That is, prove that if $A \subseteq \mathbb{N}$ is the range of a *partial* recursive function, then A is r.e., meaning that A is empty or the range of a *total* recursive function.

Exercise 2.

(a) Check that the set $\mathbb{N}[X]$ of polynomials is the underlying set (in a natural way) of a model of Robinson's Q. Deduce that Q doesn't prove that every element is even or odd.
(b) Find a model of Q in which there exists an x with $S(x) = x$.
(c) Find a model of Q in which addition is noncommutative.
(d) Find a model of Q in which multiplication is noncommutative.

Exercise 3. Let $A \subseteq \mathbb{N}$. Prove that the following are equivalent:

(i) A is r.e.
(ii) $A = W_e$ for some code e.
(iii) A is the range of an *injective* total recursive function.

Exercise 4. Prove that $A \subseteq \mathbb{N}$ is recursive iff A is finite or the range of an *increasing* total recursive function.

Exercise 5. Prove that every infinite r.e. set has an infinite recursive subset.

Exercise 6. Find a set that is Turing-reducible to K (i.e., $\leq_T K$) but isn't r.e.

Exercise 7.

(a) Use a previous exercise or argue directly that none of the following sets is recursive:

(i) $\{e \in \mathbb{N} : W_e = \emptyset\}$
(ii) $\{e \in \mathbb{N} : W_e \neq \emptyset\}$
(iii) $\{e \in \mathbb{N} : W_e = \mathbb{N}\}$
(iv) $\{e \in \mathbb{N} : W_e \neq \mathbb{N}\}$
(v) $\{e \in \mathbb{N} : |W_e| = 1\}$
(vi) $\{e \in \mathbb{N} : |W_e| \leq 4\}$
(vii) $\{e \in \mathbb{N} : |W_e| \geq 4\}$
(viii) $\{e \in \mathbb{N} : W_e \text{ is finite}\}$

(b) Which of the sets listed in part (a) are r.e.? Prove your answers.

Exercise 8. Prove that the set $\{e \in \mathbb{N} : \forall x \, \{e\}(x)\!\downarrow\}$ of codes of total recursive functions isn't even r.e.

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Exercise 9. We say that two disjoint sets $A, B \subseteq \mathbb{N}$ are **recursively inseparable** if there is no recursive set R such that $A \subseteq R$ and $B \subseteq R^c$. Show that the sets

$$A = \{e : \{e\}(e)\downarrow \text{ and } \{e\}(e) = 0\}$$

and

$$B = \{e : \{e\}(e)\downarrow \text{ and } \{e\}(e) = 1\}$$

are recursively inseparable.

Exercise 10. Show that the recursive version of Kőnig's lemma fails: there is an infinite recursive binary tree with no infinite recursive branch.