LOGIC EXERCISES – DAY 12

Exercise 1. Prove that there is a coinfinite r.e. set that meets every infinite r.e. set.

Exercise 2. We say that $E \subseteq \mathbb{N}^2$ is recursively enumerable if

$$\{\langle m, n\rangle \in \mathbb{N} \colon (m, n) \in E\}$$

is recursively enumerable. Show that if $E \subseteq \mathbb{N}^2$ is a recursively enumerable equivalence relation having finitely many equivalence classes, then E is recursive.

Exercise 3. Let $\varphi(v)$ be a formula in the language of arithmetic.

(a) Suppose that $\varphi(v)$ is Σ^0_1 (i.e. is of the form $\exists u \psi(u, v)$ where all quantifiers appearing in $\psi(u, v)$ are bounded) and that $\mathsf{PA} \vdash \exists v \varphi(v)$. Show that $\mathsf{PA} \vdash \varphi(\Delta n)$ for some numeral Δn.

(b) Give an example of a formula $\varphi(v)$ such that $\mathsf{PA} \vdash \exists v \varphi(v)$, but for all n, PA does not prove $\varphi(\Delta n)$.

(c) Suppose that $\varphi(v)$ is Σ^0_1 and T is a consistent extension of PA such that $T \vdash \exists v \varphi(v)$. Does it follow that $T \vdash \varphi(\Delta n)$ for some n?

Exercise 4 (Kleene's Second Recursion Theorem)**.** For each partial recursive function $f = f(z, \vec{x})$, there is a number z^* such that for all $\vec{x}$,

$$\varphi_{z^*}(x) = f(z^*, \vec{x}).$$

Prove this. *Hint:* Imitate the main idea in the proof of the Fixed Point Lemma.

Exercise 5. Deduce the following "fixed point" variation of Kleene's Second Recursion Theorem from our statement of the Second Recursion Theorem: For every total recursive function $f \colon \mathbb{N} \longrightarrow \mathbb{N}$, there exists $e \in \mathbb{N}$ such that $\varphi_{f(e)} = \varphi_e$.

We say that $A \subseteq \mathbb{N}$ is an *index set* if $e \in A \wedge \varphi_e = \varphi_d$ implies $d \in A$.

Exercise 6 (Rice's Theorem)**.** If $E \subseteq \mathbb{N}$ is a recursive index set, then $E = \emptyset$ or $E = \mathbb{N}$. *Hint:* Assume for a contradiction $E \neq \emptyset, \mathbb{N}$ and choose e_0 such that $\varphi_{e_0}(x) \uparrow$ for all $x \in \mathbb{N}$. If $e_0 \notin E$ then show $K \leq_1 E$. Arguing symmetrically, show $e_0 \in E$ implies $K \leq_1 \mathbb{N} \setminus E$. In either case, conclude E is not recursive, contradicting our assumption.

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Exercise 7. If T has a recursively enumerable set of axioms, then it also has a decidable set of axioms (in the same language). (What I mean here is that the set of Gödel codes of the axioms is a recursive (or r.e.) set of integers.)

Exercise 8. Let γ_{PA} be the Gödel sentence obtained by applying the Fixed Point Lemma to the formula $\neg\mathbf{Provable}_{\mathsf{PA}}$. Consider the theory $T = \mathsf{PA} \cup \{\neg\gamma_{\mathsf{PA}}\}$. Is it consistent? Is it complete? Is it sound?

We say that a set $A \subseteq \mathbb{N}$ is *immune* if it is infinite but contains no infinite recursively enumerable set. A is called *bi-immune* if both A and $\mathbb{N} \setminus A$ are immune.

Exercise 9.

(a) Show that $A = \{x \colon (\neg\exists y < x)\,[\varphi_x = \varphi_y]\}$ is immune. *Hint:* The "Fixed Point" formulation of Kleene's Second Recursion Theorem may be of some help here.
(b) Show that there are $2^{\aleph_0}$ many bi-immune sets $A \subseteq \mathbb{N}$.
(c) Show that there is a bi-immune set $A \leq_T K$.

Exercise 10. For a model $\mathbf{M}$ of PA, let $S(\mathbf{M})$ be the family of all sets $A \subseteq \mathbb{N}$ of the form

$$A = \{n \in \mathbb{N} \,:\, \mathbf{M} \models \phi(\Delta(n)^{\mathbf{M}}, \vec{a})\}$$

for some formula ϕ and some $\vec{a} \in M^k$.

(a) By using Overspill (Day 7, #6(c)) and your recursively inseparable r.e. sets (Day 11, #9) or otherwise, prove that if $\mathbf{M}$ is a nonstandard model of PA then $S(\mathbf{M})$ contains a nonrecursive set.
(b) Prove that your nonrecursive set from part (a) can actually be taken to be

$$\{n \in \mathbb{N} \,:\, \mathbf{M} \models \chi(n, x)\},$$

where $\chi(n, x)$ is a formula asserting "p_n divides x" and p_n is the n^{th} standard prime. (Hint: You needn't redo part (a).)
(c) Conclude that no nonstandard model of PA is recursive. That is, if $\mathbf{M}$ is a model of PA with underlying set $\mathbb{N}$, and $+^{\mathbf{M}}, \cdot^{\mathbf{M}} \colon \mathbb{N}^2 \to \mathbb{N}$ are recursive functions, then $\mathbf{M}$ must be the standard model $\mathbf{N}$.
(d) On the other hand, do you know a nonstandard recursive model of Robinson's Q?