LOGIC PROBLEM SET (THE LAST!) – DAY 13

Exercise 1 (Disjunctive and prenex normal forms).

(a) Prove that every $\mathcal{L}$-formula ϕ is logically equivalent (i.e., proved by the empty $\mathcal{L}$-theory to be equivalent to) a formula of the form

$$\phi \equiv Q_1 x_1 \cdots Q_n x_n \, \phi^*,$$

where ϕ^* is quantifier-free, each Q_i is either $\exists$ or $\forall$, and the free variables of ϕ^* are among the free variables of ϕ.

(b) Prove that every quantifier-free formula is logically equivalent to a formula of the form

$$\bigvee_{i \leq m} \bigwedge_{j \leq n_i} \phi_{i,j},$$

where each $\phi_{i,j}$ is an atomic formula or the negation of an atomic formula. (Think of polynomials. $\wedge$ is like multiplication and $\vee$ is like addition, and you're collecting like terms to put a polynomial in standard form. But this doesn't really help for the proof.)

Exercise 2. Prove that every infinite poset contains an infinite chain or infinite antichain.

Exercise 3. Let F be a field and let $\mathcal{L}$ be the language of F-vector spaces: it has a binary function symbol $+$, a unary function symbol $-$, a constant symbol 0, and for each $\alpha \in F$ a unary function symbol μ_α for scalar multiplication by α. (So elements of the structure will be vectors — you don't have scalars *and* vectors with relations distinguishing them.) Show that the $\mathcal{L}$-theory of infinite F-vector spaces admits qe. Deduce that this theory is complete.

Exercise 4. Prove that DLO admits qe and is complete.

Exercise 5.

(a) Show that $(\mathbb{R}, 0, +)$ admits qe.

(b) Show that $(\mathbb{Q}, 0, +)$ is an elementary substructure of $(\mathbb{R}, 0, +)$.

(c) Is $(\mathbb{Q}, 0, +, \cdot)$ an elementary substructure of $(\mathbb{R}, 0, +, \cdot)$?

Exercise 6 (Reflection plus compactness doesn't imply Con(ZFC)).

Theorem (Reflection, weak form). For every sentence σ in the language of set theory, the following is a theorem of ZFC: there is a set M (in fact, M can be taken to be V_α for some α) such that $(M, \in) \models \sigma$ iff $V \models \sigma$, that is, iff σ is true in the universe.

Since ZFC is true in the universe, for every sentence of ZFC there is a set model of that sentence. It follows that every finite subset of ZFC has a set model. Explain this, and explain why it does *not* follow from Compactness that ZFC $\vdash$ Con(ZFC). (Good thing, since that would contradict Kurt's Second Incompleteness Theorem.)

How might you prove Reflection?

Exercise 7. Let K be a field and let $\overline{K}$ be the algebraic closure of K. A nonconstant polynomial $f \in K[x_1, \ldots, x_n]$ is called **irreducible** if whenever $f = gh$ for some $g, h \in K[x_1, \ldots, x_n]$, either $\deg(g) = 0$ or $\deg(h) = 0$. Furthermore, f is called **absolutely irreducible** if it is irreducible over the algebraic closure, i.e., in $\overline{K}[x_1, \ldots, x_n]$.

Date: Wednesday, 8 July 2015.

E.g. the polynomial $x^2 + 1 \in \mathbb{R}[x]$ is irreducible, but it is not absolutely irreducible, since $x^2 + 1 = (x + i)(x - i)$ in $\mathbb{C}[x]$. On the other hand, $xy - 1 \in \mathbb{Q}[x, y]$ is absolutely irreducible.

Let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ and prove the following:

Theorem (Noether–Ostrowski Irreducibility Theorem)**.** For $f \in \mathbb{Z}[x_1, \ldots, x_n]$ and a prime p, let f_p denote the polynomial in $\mathbb{F}_p[x_1, \ldots, x_n]$ obtained by applying the canonical map $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ to the coefficients of f (i.e., modding out the coefficients by p). For all $f \in \mathbb{Z}[x_1, \ldots, x_n]$, f is absolutely irreducible (as an element of $\mathbb{Q}[x_1, \ldots, x_n]$) iff f_p is absolutely irreducible (as an element of $\mathbb{F}_p[x_1, \ldots, x_n]$) for cofinitely many p.

Hint: Your proof should be shorter than the statement of the problem. Apparently the original algebraic proof was fairly complicated.

Exercise 8 (random graph)**.** Let σ_n be the assertion that, if X and Y are disjoint sets of vertices both of cardinality $\leq n$, then there is a vertex x not in $X \cup Y$ adjacent to every member of X and to no member of Y. Let $T = \{\sigma_n : n \in \mathbb{N}\}$.

(a) Convince yourself that σ_n can be written as a sentence in the language of graphs.
(b) Prove that there is a countable model of T and that any two countable models of T are isomorphic. (This is a set of axioms for the random graph.)
(c) Prove that the random graph (i.e., a countable model of the axioms σ_n above) includes every finite graph as a full subgraph. (Recall that G is a **full** subgraph of H if the embedding $G \to H$ is a strong homomorphism.)
(d) Prove that the theory of the random graph admits qe.

This theory axiomatizes the "almost-sure" theory of finite graphs. That is, every sentence in the language of graphs is, among finite graphs, asymptotically almost-surely true or almost-surely false, according to whether it's true in the random graph. So this almost-sure theory is decidable and complete.

Exercise 9.

(a) Show that $\mathrm{Th}(\mathbb{Z}, S)$ has quantifier elimination. Here $S(x) = x + 1$.
(b) Show that $\mathrm{Th}(\mathbb{N}, S)$ doesn't have quantifier elimination.

Exercise 10. An abelian group is **torsion-free** if the identity element is the only element of finite order. An abelian group $(A, +)$ is **divisible** if every element $a \in A$ is an n^{th} multiple, for every n. That is, A is divisible if for every $a \in A$ and every $n \in \mathbb{N}$ there is $b \in A$ such that

$$nb = \underbrace{b + \cdots + b}_{n \text{ terms}} = a.$$

(a) Suppose that G and H are nontrivial torsion-free divisible abelian groups, $G \subseteq H$, $\psi(\vec{v}, w)$ is quantifier-free, $\vec{a} \in G$, $b \in H$, and $H \models \psi(\vec{a}, b)$. Prove that there is $c \in G$ such that $G \models \psi(\vec{a}, c)$.
(b) Use the following fact about torsion-free abelian groups to prove that the theory of divisible abelian groups (in the language $\{0, +, -\}$, so substructures of groups are subgroups) has q.e.

Theorem. Let G be a torsion-free abelian group. There is a torsion-free abelian group G^*, called the **divisible hull** of G, and an embedding $i\colon G \to G^*$ such that if $j\colon G \to H$ is any embedding of G into a torsion-free abelian group H, then there is $h\colon G^* \to H$ such that $j = h \circ i$.