LOGIC EXERCISES – DAY 2

Exercise 1.

(a) Find a sentence true in the structure $(\mathbb{N}, <)$ but not in the structure $(\mathbb{Z}, <)$.
(b) Find a sentence true in the structure $(\mathbb{Q}, <)$ but not in the structure $(\mathbb{Z}, <)$.
(c) Can you think of a sentence true in $(\mathbb{Q}, <)$ but false in $(\mathbb{R}, <)$?

Exercise 2. Define appropriate languages for

(a) vector spaces over $\mathbb{Q}$ (better yet, over any field);
(b) metric spaces.

Is there a language suitable to describe topological spaces? Explain your answer.

Exercise 3. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a function. Prove that the set of points at which f is continuous is definable in the structure $(\mathbb{R}, <, f)$.

Exercise 4. A formula is called *universal* if it is of the form $\forall \vec{x}\, \psi$, where ψ is quantifier-free. *Existential* formulas are defined similarly (with $\exists$ instead of $\forall$). Let $\mathbf{A} \subseteq \mathbf{B}$ be $\mathcal{L}$-structures, $\phi = \phi(v)$ be an $\mathcal{L}$-formula, and $a \in A^n$. Show that

(a) If ϕ is quantifier-free, then $\mathbf{A} \models \phi(a) \iff \mathbf{B} \models \phi(a)$.
(b) If ϕ is universal, then $\mathbf{B} \models \phi(a) \Longrightarrow \mathbf{A} \models \phi(a)$.
(c) If ϕ is existential, then $\mathbf{A} \models \phi(a) \Longrightarrow \mathbf{B} \models \phi(a)$.

Exercise 5. Let S be a finite set of vectors in $\mathbb{Q}^d$. Show that S is linearly independent over $\mathbb{Q}$ iff it is linearly independent over $\mathbb{R}$. (Hint: previous exercise)

Exercise 6. Show that any definable set in $\mathbf{N}$ is 0-definable.

Exercise 7. Let $(A, 0, +)$ be a non-trivial abelian group, and let $R^{\mathbf{A}}$ be the relation on A defined by

$$R^{\mathbf{A}}(x, y, z, w) \iff x + y = z + w$$

Show that the addition map $(x, y) \longmapsto x + y\colon A \times A \longrightarrow A$ is definable in the structure $\mathbf{A} = (A, R^{\mathbf{A}})$ from the parameter 0, but is not definable without parameters.

Exercise 8. Determine whether the following are 0-definable.

(a) The set $\mathbb{N}$ in $(\mathbb{Z}, +, \cdot)$. (Hint: You may need a nontrivial fact from elementary number theory.)
(b) The set of nonnegative numbers in $(\mathbb{Q}, +, \cdot)$.
(c) The set of nonnegative numbers in $(\mathbb{Q}, +)$.
(d) The set of positive numbers in $(\mathbb{R}, <)$.
(e) The function $\max(x, y)$ in $(\mathbb{R}, <)$.
(f) The function $\text{average}(x, y)$ in $(\mathbb{R}, +, \cdot)$.
(g) The function $\text{average}(x, y)$ in $(\mathbb{R}, <)$.
(h) 2 in $(\mathbb{R}, +, \cdot)$.
(i) The relation $d(x, y) \leq 2$ in a graph, where d denotes the distance function.
(j) The relation $d(x, y) = 2$ in a graph (as above).

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Exercise 9. Suppose that $(X, <)$ and $(Y, <)$ are countably infinite linear orders that are dense, meaning that between any two distinct points there is a third point. (If $x < y$ then there is z in the interval (x, y).) Suppose also that there is no minimum or maximum in either of these orderings. Prove that $(X, <)$ and $(Y, <)$ are isomorphic.