LOGIC EXERCISES – DAY 3

Exercise 1. Is there $\mathbf{A} \subseteq \mathbf{B}$ such that A is isomorphic to B but the inclusion map is not an elementary embedding?

Exercise 2. Define theories for vector spaces over $\mathbb{Q}$ and for metric spaces. (See Exercise 2 from Day 2.)

Exercise 3.

(a) An $\mathcal{L}$-theory T is called **absolutely categorical** if it is satisfiable and any two models of T are isomorphic. Show that if T is absolutely categorical, then every model of T is finite.

(b) Give an example of a *finite* $\mathcal{L}$-theory T all of whose models are infinite.

(c) Show that there is an $\mathcal{L}_{\text{group}}$-sentence ϕ such that

$$\mathbf{M} \models \phi \iff \mathbf{M} \cong \mathbb{Z}/2\mathbb{Z}.$$

(d) More generally, let $\mathcal{L}$ be a finite language and $\mathbf{A}$ be a finite $\mathcal{L}$-structure. Show that there is an $\mathcal{L}$-sentence ϕ such that for any $\mathcal{L}$-structure $\mathbf{B}$,

$$\mathbf{B} \models \phi \iff \mathbf{B} \cong \mathbf{A}.$$

In particular,

$$\mathbf{B} \equiv \mathbf{A} \iff \mathbf{B} \cong \mathbf{A}.$$

Exercise 4. Let $\mathbf{C}_{\text{exp}} = (\mathbb{C}, 0, 1, +, \cdot, \exp)$, where $\exp\colon \mathbb{C} \to \mathbb{C}$ is the usual exponentiation $z \mapsto e^z$. Show that $\mathbb{Z}$ is definable in $\mathbf{C}_{\text{exp}}$. Conclude that $\mathbb{N}$ is too.

Exercise 5.

(a) Let $\mathbf{A} \subseteq \mathbf{B}$ and assume that for any finite $S \subseteq A$ and any $b \in B$ there exists an automorphism f of $\mathbf{B}$ that fixes S pointwise (i.e., $f(a) = a$ for every $a \in S$) and $f(b) \in A$. Show that $\mathbf{A} \prec \mathbf{B}$.

(b) Give an example to show that the converse of part (a) can fail. That is, find structures $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{A} \prec \mathbf{B}$ but it isn't the case that for every finite set S and every $b \in B$ there's an automorphism. . . .

Exercise 6. Show that $(\mathbb{Q}, <) \prec (\mathbb{R}, <)$. (Hint: previous problem) Conclude that $(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$ but $(\mathbb{Q}, <) \not\cong (\mathbb{R}, <)$. (Also, see Problem 1(c) from Day 2.)

Exercise 7. Let $\mathcal{L}$ be a language whose only symbol is a unary function symbol f. For each $\mathcal{L}$-sentence σ, let $\text{Spec}(\sigma)$ be the finite spectrum of σ, i.e., the set of all cardinalities of finite models of σ. Let $a \geq 0$ and $b \geq 1$ be integers. Give an $\mathcal{L}$-sentence σ such that

$$\text{Spec}(\sigma) = \{a + bn \colon n \in \mathbb{N}\}$$

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Exercise 8.

(a) Let $\mathcal{L} = \{E\}$, where E is a binary relation symbol, and let ϕ be the $\mathcal{L}$-sentence asserting that E is an equivalence relation whose classes all have exactly 2 elements. Show that the finite spectrum of ϕ is all positive even integers.

(b) For each of the following subsets of $\mathbb{N}$, show that it is the finite spectrum of some sentence ϕ in some language $\mathcal{L}$:

(i) $\{2^n : n \in \mathbb{N}\}$
(ii) $\{2^n 3^m : n, m \in \mathbb{N}\}$
(iii) $\{n^2 : n \in \mathbb{N}\}$
(iv) $\{n \in \mathbb{N} : n \text{ is composite}\}$
(v) $\{p^n : p \text{ is prime and } n \in \mathbb{N}\}$
(vi) $\{p : p \text{ is prime}\}$.
(vii) $\{\binom{n}{2} : n \in \mathbb{N}\}$.

Open Question: Must the complement of a finite spectrum be a finite spectrum too?