LOGIC EXERCISES – DAY 4

Exercise 1. In lecture we deduced the compactness theorem from the ultrafilter lemma (every filter extends to an ultrafilter). Without using Choice, use the compactness theorem to prove that every filter extends to an ultrafilter.

Exercise 2.

(a) A structure $\mathbf{A} = (A, \{f_k\}_{k\in\mathbb{N}})$ is called a *Jónsson algebra* iff it has no proper substructures of the same cardinality. Show there exists a Jónsson algebra of cardinality $\aleph_0$.

(b) Let $\{A_\alpha \colon \alpha < \kappa\}$ be a collection of sets indexed by some cardinal κ such that each A_α has cardinality κ. Show that there is a sequence $\{a_\alpha \colon \alpha < \kappa\}$ of *distinct* elements such that $a_\alpha \in A_\alpha$ for all $\alpha < \kappa$. (You may use the Axiom of Choice, but note that we don't assume any disjointness properties of the A_α's).

(c) Let $\{A_\alpha \colon \alpha < \kappa\}$ be as before. Show that there exists a sequence $\{B_\alpha \colon \alpha < \kappa\}$ of *pairwise disjoint* sets such that $B_\alpha \subseteq A_\alpha$ and each B_α has cardinality κ (Hint: use the fact that there is a well-ordering of $\kappa \times \kappa$ of order-type κ).

(d) Recall that the GCH (Generalized Continuum Hypothesis) is the following statement: $\forall\kappa\,(2^\kappa = \kappa^+)$. Recall that this implies $|[\kappa^+]^\kappa| = \kappa^+$, where $[\kappa^+]^\kappa$ denotes the collection of all subsets of κ^+ of size κ. Enumerate this collection as $\{X_\alpha \colon \alpha < \kappa^+\}$. Use part (c) to define a function $f \colon \kappa^+ \times \kappa^+ \longrightarrow \kappa^+$ such that

$$\kappa \leq \alpha < \kappa^+ \wedge \beta < \alpha \implies \text{the image of } f \restriction X_\beta \times \{\alpha\} \text{ includes } \alpha \text{ as a subset}$$

(e) Show that (κ^+, f) is a Jónsson algebra. Conclude that the GCH implies there exists a Jónsson algebra on every successor cardinal. (In fact, it can be shown that there is a Jónsson algebra on each $\aleph_n$, $n < \omega$ even without invoking the GCH. Similarly, it can be shown that there is a Jónsson algebra on $\aleph_{\omega+1}$ without invoking the GCH, as was demonstrated by Shelah.)

Exercise 3.

(a) Let $\mathcal{L} = \{0, +\}$. Show that $\mathbf{R} = (\mathbb{R}, 0, +)$ *admits quantifier elimination*, i.e. show that for any $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$, there is a quantifier-free $\mathcal{L}$-formula $\psi(x_1, \ldots, x_n)$ such that

$$\mathbf{R} \models \forall x_1, \ldots, \forall x_n(\varphi(x_1, \ldots, x_n) \leftrightarrow \psi(x_1, \ldots, x_n))$$

(b) Show that $(\mathbb{Q}, 0, +)$ is an elementary substructure of $(\mathbb{R}, 0, +)$.

(c) Is $(\mathbb{Q}, 0, +, \cdot)$ an elementary substructure of $(\mathbb{R}, 0, +, \cdot)$?

Exercise 4. Prove that if a theory has arbitrarily large finite models, then it has an infinite model. Conclude that the class of finite structures (in any language) is not axiomatizable.

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Exercise 5.

(a) Prove that the class of bipartite graphs is axiomatizable (in the language of graphs).
(b) Find an expansion of the language of graphs in which the class of bipartite graphs is finitely axiomatizable.
(c) Prove that the class of bipartite graphs is not finitely axiomatizable in the language of graphs, perhaps by considering an ultraproduct of odd cycles.

Exercise 6. Notice that complex conjugation $z \mapsto \overline{z}$ is an automorphism of the structure $\mathbf{C} = (\mathbb{C}, 0, 1, +, \cdot)$. It's an *involution*, meaning that it's its own inverse.

(a) Prove that the singleton $\{i\}$ is not 0-definable in $\mathbf{C}$.
(b) Prove that the imaginary-part function $z \mapsto \mathfrak{Im}(z)$ (i.e., $a + bi \mapsto b$) is not 0-definable in $\mathbf{C}$.
(c) Prove that if one of the following is 0-definable in $\mathbf{C}$, then all three of them are:
    (i) the set $\mathbb{R}$
    (ii) the real-part function $z \mapsto \mathfrak{Re}(z)$ (i.e., $a + bi \mapsto a$)
    (iii) the modulus-squared function $z \mapsto |z|^2$.
(d) $(*)$ Are the three functions/relations above 0-definable in $\mathbf{C}$ or not?

Exercise 7.

(a) Find a structure and an automorphism f of it that isn't an involution, meaning $f \circ f$ is not the identity.
(b) More: provide an example of an infinite structure $\mathbf{A}$ and an automorphism f of it that *acts transitively* in the following sense: for every $x \in A$ and every $y \in A$ there is $n \in \mathbb{Z}$ such that $f^n(x) = y$. (If $n < 0$, then f^n denotes the n-fold iteration of f^{-1}.)
(c) Can you think of an $\mathcal{L}$-structure $\mathbf{A}$ such that *every* bijection $A \to A$ is an automorphism of $\mathbf{A}$? Call such an $\mathcal{L}$-structure **outrageously homogeneous**. Can you give necessary and sufficient conditions on $\mathcal{L}$ for there to exist an outrageously homogeneous $\mathcal{L}$-structure?
(d) For a language $\mathcal{L}$ satisfying your necessary and sufficient conditions from (c), give an axiomatization of the class of infinite outrageously homogeneous $\mathcal{L}$-structures.

Exercise 8.

(a) Prove that if $\mathbf{A}_0 \preceq \mathbf{A}_1 \preceq \mathbf{A}_2 \dots$ is a chain of elementary substructures, then $\mathbf{A}_\omega = \bigcup_{n\in\mathbb{N}} \mathbf{A}_n$ (defined in the natural way) is an elementary superstructure of each element in the chain.
(b) You should be familiar with the notion of "club" from Sherwood's class. Prove

$$\{\alpha < \omega_1 \colon (\alpha, \in) \preceq (\omega_1, \in)\}$$

is a club in ω_1.

Exercise 9. (The following is Problem 4.2.3 of *Introduction to Cardinal Arithmetic by Holz, Steffens, and Weitz*):

(a) Show that, for any ordinals α and β such that α is a successor and β is a limit ordinal, $(\alpha, \in)$ is not elementarily equivalent to $(\beta, \in)$.
(b) Assume that β is a limit ordinal with $\beta > \omega$. Show that $(\omega, \in)$ is not elementarily equivalent to $(\beta, \in)$.
(c) Show that $(\omega \cdot \omega, \in)$ is not elementarily equivalent to $(\omega_1, \in)$.
(d) Show that $(\omega^\omega, \in)$ is not elementarily equivalent to $(\omega_1, \in)$.