LOGIC EXERCISES – DAY 5

Exercise 1. Don't use the Completeness Theorem for any part of this problem.

(a) Prove that every satisfiable $\mathcal{L}$-theory is included in some larger $\mathcal{L}$-theory that is semantically complete and satisfiable.

(b) Prove that every consistent $\mathcal{L}$-theory is included in some larger $\mathcal{L}$-theory that is complete and consistent.

Exercise 2 (Important)**.** A class of $\mathcal{L}$-structures is finitely axiomatizable iff it and its complement are both axiomatizable.

Exercise 3. Prove Ramsey's theorem (the infinite version) for arbitrary (finite) arity of colorings and for any (finite) number of colors. That is, prove that for any $k, r \in \mathbb{N}$, if $[\mathbb{N}]^k$ is r-colored, then there is an infinite monochromatic set. (Hint: You don't need to reprove the theorem.) Verify that the compactness argument from lecture goes through with 2s changed to ks or rs appropriately, so that you get a general version of the finite version of Ramsey's theorem.

Exercise 4. Let $\mathbf{A}$ be an infinite structure in some language. Prove that the class of structures isomorphic to $\mathbf{A}$ is not axiomatizable.

Exercise 5. Prove that a theory is semantically complete iff all of its models are elementarily equivalent.

Exercise 6. DLO is the theory of dense linear orders in the language $\{\leq\}$ of orderings. Recall from an earlier problem set that any two countable dense linear orders without endpoints (minima or maxima) are isomorphic. Describe all countable models of DLO.

Exercise 7 (ultrapower)**.** Let $\mathbf{M}$ be an $\mathcal{L}$-structure, and let U be a nonprincipal ultrafilter on ω. The **ultrapower** $\mathbf{M}^\omega/U$ of $\mathbf{M}$ by U is the ultraproduct $\prod_{n\in\omega} \mathbf{M}_n/U$, where $\mathbf{M}_n = \mathbf{M}$ for every n. Show that the map

$$\iota\colon \mathbf{M} \to \mathbf{M}^\omega/U, \quad \iota(a) = [x \mapsto a]_U$$

is an elementary embedding and is surjective iff M is finite.

Exercise 8. Prove that a graph is n-colorable iff each of its finite subgraphs is n-colorable.

Exercise 9. Let X be an uncountable set, and suppose that $[X]^2$ is 2-colored. Must there be an uncountable monochromatic set?

Exercise 10.

(a) Prove that there is a family F of $\mathfrak{c}(= 2^{\aleph_0})$ functions $\mathbb{N} \to \mathbb{R}$ such that any distinct $f, g \in F$ agree on only a finite set. (Don't over-think it.)

(b) Prove that there is a family of $\mathfrak{c}$-many functions $\mathbb{N} \to \mathbb{N}$ any two of which agree on only a finite set. (You might try to use your answer to the previous part, or you might find it easier to argue directly.)

(c) Suppose that we have $\mathcal{L}$-structures $\mathbf{A}_n$ for $n \in \mathbb{N}$ (that may be finite or infinite), and suppose that there is no finite bound on the sizes of the underlying sets A_n. (That is, suppose that for every $k \in \mathbb{N}$ there is n such that $|A_n| > k$.) Let U be a nonprincipal ultrafilter on $\mathbb{N}$. Prove that the size of the ultraproduct $\prod_{n\in\mathbb{N}} \mathbf{A}_n/U$ is at least $\mathfrak{c}$.

Exercise 11. Prove that the class of wellorders is not axiomatizable in the language $\{\leq\}$ of orderings. Prove that its complement, the class of non-wellorders ("illorders"?), isn't axiomatizable either.

Exercise 12. Let $T = \mathrm{Th}(\mathbb{Z}, <)$. Show that there is a model $\mathbf{M}$ of T with an order-preserving embedding $\mathbb{Q} \to M$.

Exercise 13. Let T be the theory of $\mathbf{A} = (\mathbb{Z}, S)$, where $S^{\mathbf{A}}(n) = n+1$. Show that up to isomorphism there are exactly $\aleph_0$ many countably models of T, and that T is categorical in every uncountable cardinality.

Exercise 14. Prove that the class of connected graphs is not axiomatizable (in the language of graphs). Prove that the class of disconnected graphs isn't axiomatizable either.

Exercise 15. Let $\mathfrak{N}$ be a nonstandard model of Peano Arithmetic. Show that there is an element $a \in \mathfrak{N}$ such that for any standard prime number p, p^{17} divides a but p^{18} does not.