LOGIC EXERCISES – WEEK 2, DAY 1

Exercise 1. Let DLO be the theory of dense linear orders *without endpoints* in the language $\mathcal{L} = \{<\}$ of orderings. On a previous homework we showed DLO is $\aleph_0$-categorical, i.e. any two models of DLO of cardinality $\aleph_0$ are isomorphic. Is DLO $\aleph_1$-categorical?

Exercise 2. Prove that a class of $\mathcal{L}$-structures is axiomatizable iff it's closed under elementary equivalence and taking ultraproducts. Conclude that a class of $\mathcal{L}$-structures is finitely axiomatizable iff both it and its complement are closed under elementary equivalence and taking ultraproducts.

Exercise 3. Write down axioms — in a suitable language — for the theory of groups with an element of infinite order. Can this be done in the language of groups?

Exercise 4. Let IC be the sentence in the language of set theory denoting "there exists an inaccessible cardinal." Prove (informally):

(a) $\mathsf{ZFC} + \mathsf{IC} \vdash \mathrm{Con}(\mathsf{ZFC})$.

(b) $\mathsf{ZFC} + \mathsf{IC} \vdash \mathrm{Con}(\mathsf{ZFC} + \mathrm{Con}(\mathsf{ZFC}))$

($\mathrm{Con}(T)$ is the statement in the language of set theory asserting that the theory T is consistent.)

Exercise 5. Let I be an infinite set.

(a) Let κ be an infinite cardinal. A filter $\mathcal{F}$ over I is said to be κ-regular if there exists $\mathcal{E} \subseteq \mathcal{F}$ with $|\mathcal{E}| = \kappa$ such that each $i \in I$ belongs to only finitely many elements of $\mathcal{E}$. Show that a filter $\mathcal{F}$ on I is $\aleph_0$-regular if and only if there exists a countable descending chain $I = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$ of elements $I_n \in \mathcal{F}$ such that $\bigcap_n I_n = \emptyset$.

(b) A filter $\mathcal{F}$ is said to be *countably incomplete* iff there exists countable $\mathcal{E} \subseteq \mathcal{F}$ such that $\bigcap \mathcal{E} \notin \mathcal{F}$. Prove that an *ultra*filter $\mathcal{U}$ on I is $\aleph_0$-regular iff it is countably incomplete.

(c) Give an example of a filter on some set which is countably incomplete but *not* $\aleph_0$-regular.

(d) Prove that if $|I| = \kappa \geq \aleph_0$, then there exists a κ-regular ultrafilter $\mathcal{U}$ over I.

Exercise 6. Suppose that $\{\mathbf{A}_i \colon i \in I\}$, $\{\mathbf{B}_i \colon i \in I\}$ are collections of $\mathcal{L}$-structures and $\mathcal{U}$ is an ultrafilter on I.

(a) Prove that if $|\mathbf{A}_i| \leq |\mathbf{B}_i|$ for all $i \in I$, then $\left|\prod_{i\in I} \mathbf{A}_i/\mathcal{U}\right| \leq \left|\prod_{i\in I} \mathbf{B}_i/\mathcal{U}\right|$.

(b) Prove $\min_{i\in I} |\mathbf{A}_i| \leq \left|\prod_{i\in I} \mathbf{A}_i/\mathcal{U}\right| \leq \left|\prod_{i\in I} \mathbf{A}_i\right|$.

(c) Recall the definition of *κ-regular* from the previous exercise. Show that if $\mathcal{U}$ is a κ-regular ultrafilter on a set I of cardinality $\kappa \geq \aleph_0$ and $\mathbf{A}$ is an infinite $\mathcal{L}$-structure then we have: $\left|\prod_{i\in I} \mathbf{A}/\mathcal{U}\right| = |\mathbf{A}|^{\kappa}$.

Exercise 7. Prove the following slightly stronger version of finite Ramsey. For every m there is an n with the following property: whenever $[n]^2$ is 2-colored there is a monochromatic set M of size $\geq m$ satisfying $|M| > \min M$.

The Paris–Harrington theorem says that this statement is not a theorem of Peano Arithmetic. It's proved by showing that (in PA) this strong form of Ramsey's theorem implies the consistency of PA.

Exercise 8. Given an example of languages $\mathcal{L} \subseteq \mathcal{L}'$ as well as theories $T \subseteq T'$ such that T is a complete $\mathcal{L}$-theory, T' is a complete $\mathcal{L}'$-theory, and T' is $\aleph_1$-categorical but T is not.