LOGIC EXERCISES – WEEK 2, DAY 2

Exercise 1. An $\mathcal{L}$-theory A is an **axiomatization** of an $\mathcal{L}$-theory T if for every $\mathcal{L}$-sentence σ,

$$A \models \sigma \iff T \models \sigma.$$

A theory is called **finitely axiomatizable** if it has a finite axiomatization. Prove that an $\mathcal{L}$-theory T is finitely axiomatizable iff there is a finite *subset* of T that is an axiomatization of T.

Exercise 2. Prove that the following are not finitely axiomatizable:

(a) The theory of fields of characteristic 0.
(b) ACF.
(c) ACF_0.
(d) ACF_p.

Exercise 3 (multiple colorings)**.**

(a) If $\chi_1, \dots, \chi_n : [\mathbb{N}]^2 \to 2$ are finitely many 2-colorings, then there is a single infinite set that is monochromatic for all of the χ_k.
(b) On the other hand, if you're given (countably) infinitely many 2-colorings of $[\mathbb{N}]^2$, you may not be able to find an infinite set simultaneously monochromatic for all of them. Find an example.
(c) On the third hand, prove: if you have countably many 2-colorings of $[\mathbb{N}]^2$, then there is a single infinite set X simultaneously *almost* monochromatic for all of them. (An infinite set X is **almost monochromatic** for χ if there is some finite set F such that $X \setminus F$ is monochromatic for χ.)

Exercise 4. What if we color all finite subsets of $\mathbb{N}$ instead of just the pairs?

(a) Show that there is a 2-coloring of $[\mathbb{N}]^{<\omega}$ with no infinite monochromatic set. That is, find a coloring $\chi : [\mathbb{N}]^{<\omega} \to 2$ such that there is no infinite $M \subseteq \mathbb{N}$ with $\chi \restriction [M]^{<\omega}$ constant.
(b) You can't even hope for a single set X such that for every n every n-element subset of X has the same color. Prove this by finding an example.
(c) BUT! Use (your proof of) part (c) of the previous problem to prove that if $\chi : [\mathbb{N}]^{<\omega} \to 2$ is a coloring, then there is an infinite set $X \subseteq \mathbb{N}$ such that for every k, X is almost monochromatic for the coloring $\chi \restriction [\mathbb{N}]^k$.

Exercise 5. Prove that the class of wellorders is not axiomatizable in the language $\{\leq\}$ of orderings. Prove that its complement, the class of non-wellorders ("illorders"?), isn't axiomatizable either.

Exercise 6.

(a) Show that for any model $\mathbf{M}$ of PA, there is a unique homomorphism $f : \mathbf{N} \to \mathbf{M}$ and this f is injective. (That is, f is an $\mathcal{L}_a$-embedding.)

(b) Continuing part (a), define the standard part of $\mathbf{M}$ by

$$\overline{\mathbb{N}} = f[\mathbb{N}].$$

Show that if $\mathbf{M}$ is nonstandard then $\overline{\mathbb{N}}$ is not definable in $\mathbf{M}$.

(c) (Overspill) Let $\mathbf{M}$ be a nonstandard model of PA, let $\phi(x, \vec{y})$ be an $\mathcal{L}_a$-formula, where $\vec{y}$ is a k-vector, and $\vec{a} \in M^k$. Show that if $\mathbf{M} \models \phi(n, \vec{a})$ for infinitely many $n \in \overline{\mathbb{N}}$, then there is $w \in M \setminus \overline{\mathbb{N}}$ such that $\mathbf{M} \models \phi(w, \vec{a})$. In other words, if a statement is true about infinitely many natural numbers, then it is true about an infinite number.

Exercise 7. Let κ be an infinite cardinal. Prove that there are 2^κ non-isomorphic linear orders of size κ, perhaps by replacing some points in κ with copies of $\mathbb{Q}$.