LOGIC EXERCISES – DAY 8

Exercise 1. Prove the "Definition by cases" part of the lemma from lecture today.

Exercise 2. Prove that the class of noetherian rings is not axiomatizable (in the language of rings).

Exercise 3. If a polynomial map $\mathbb{C}^n \to \mathbb{C}^n$ is injective, then it's surjective too. (Hint: it's true in finite fields. . .)

Exercise 4 (infinite-exponent Ramsey)**.** You might wonder whether Ramsey's theorem holds for colorings of the infinite subsets of $\mathbb{N}$. Prove that it doesn't. That is, show that there is a coloring $\chi\colon [\mathbb{N}]^\omega \to 2$ such that no infinite $X \subseteq \mathbb{N}$ is monochromatic. (An infinite set $\chi \upharpoonright [X]^\omega$ is constant.) Notice anything about your proof?

Exercise 5. Let X be a set (probably infinite) of prime numbers. Show that there is a model $\mathbf{A}$ of arithmetic and an element $a \in A$ whose prime divisors are exactly the elements of X. Conclude that there are $2^{\aleph_0}$ countable models of arithmetic up to isomorphism. (You may take "arithmetic" here to mean PA or Th($\mathbf{N}$); whatever.)

Exercise 6. For any two formulas $\chi_1 = \chi_1(v_0, v_1)$ and $\chi_2 = \chi_2(v_0, v_1)$ in the language of arithmetic, there are sentences σ and τ of arithmetic such that

$$\mathbf{N} \models \sigma \leftrightarrow \chi_1(\ulcorner\sigma\urcorner, \ulcorner\tau\urcorner)$$
$$\mathbf{N} \models \tau \leftrightarrow \chi_2(\ulcorner\sigma\urcorner, \ulcorner\tau\urcorner).$$

Exercise 7. Let $X \subseteq \mathbb{R}^2$ be an infinite collection of points in *standard position*, i.e. no three of which are collinear. Prove that there is an infinite subcollection $X_0 \subseteq X$ such that no point in X_0 lies in the interior of a triangle formed by 3 other points in X_0. Hint: Ramsey.