LOGIC EXERCISES – DAY 9

Exercise 0. Solve leftover problems from yesterday's problem set.

Exercise 1. Suppose that $f\colon \mathbb{N} \to \mathbb{N}$ is a recursive function such that $f(n) > n$ for all $n \in \mathbb{N}$. Prove that its image $f[\mathbb{N}]$ is a recursive subset of $\mathbb{N}$.

Exercise 2 (FERMAT'S LAST THM MOD P, OMGOMG!)**.**

(a) Fix $r \in \mathbb{N}$. Prove that there is $n \in \mathbb{N}$ such that whenever $n = \{0, \ldots, n-1\}$ is r-colored (the elements are colored, not pairs), there is a monochromatic solution to $x + y = z$, i.e., a pair of elements $x, y < n$ such that $\{x, y, z\}$ is a monochromatic set. (Hint: You needn't do it from scratch.)

(b) Prove, using part (a) if you want, that for every n the equation $x^n + y^n = z^n$ has a nontrivial solution mod p for all sufficiently large p.

Exercise 3. Consider the Fibonacci function F defined by the following declaration:

$$F(0) = 1, \quad F(1) = 1, \quad F(n+2) = F(n) + F(n+1).$$

This function is not *prima facie* primitive recursive; that is, the declaration above doesn't define it by primitive recursion from primitive recursive functions. Show that F is nonetheless primitive recursive.

Exercise 4. Our analysis of Gödel's β-function in lecture gave the following. If $B(N) = \prod_{i<n}(1 + (1+i)N!)$, then for every $n \in \mathbb{N}$ and $\vec{a} \in \mathbb{N}^n$,

$$\text{whenever } N \geq \max\{N, a_0, \ldots, a_{n-1}\},$$
$$\text{there is } a < B(N) \text{ such that } (\forall i < n)\beta(a, i) = a_i.$$

Use this to prove that the following sequence-coding operations are primitive recursive:

(i) $(a_0, \ldots, a_{n-1}) \mapsto \langle a_0, \ldots, a_{n-1} \rangle$ (for fixed n, an n-ary function)
(ii) $a \mapsto \mathrm{lh}(a)$
(iii) $(a, i) \mapsto (a)_i$
(iv) $(a, i) \mapsto \mathrm{InitSeg}(a, i)$
(v) $(a, b) \mapsto a * b$.

(Remember that these are operations *on the level of codes*, so for example if a codes a sequence, then $\mathrm{lh}(a)$ is the length of the sequence that a codes.)

Exercise 5. Show that the following functions and relations are primitive recursive:

(a) $(x, y) \mapsto \operatorname{quot}(x, y)$.
(b) $(x, y) \mapsto \operatorname{rem}(x, y)$.
(c) $\{n \in \mathbb{N} : n \text{ is prime}\}$.
(d) $p(i) =$ the i^{th} prime number.

(Here $\operatorname{quot}(x, y) = q$ and $\operatorname{rem}(x, y) = r$, where (q, r) is the unique pair of natural numbers such that $x = qy + r$ and $0 \leq r < y$. Make some sensible choice when $y = 0$.)

Exercise 6. Prove the following facts about the Ackermann function:

(a) $A(n, x + y) \geq A(n, x) + y$.
(b) $n \geq 1 \Longrightarrow A(n + 1, y) > A(n, y) + y$.
(c) $A(n + 1, y) \geq A(n, y + 1)$.
(d) $2A(n, y) < A(n + 2, y)$.
(e) $x < y \Longrightarrow A(n, x + y) \leq A(n + 2, y)$.

Exercise 7. Show that the graph of the Ackermann function is primitive recursive. Deduce that the Ackermann function is recursive.

Exercise 8. Show that the Ackermann function grows faster than any primitive recursive function. More precisely, prove that for any primitive recursive function $f \colon \mathbb{N}^k \to \mathbb{N}$, there exists $n_f \in \mathbb{N}$ such that $f(\vec{x}) \leq A(n_f, |\vec{x}|_1)$ for all $\vec{x} \in \mathbb{N}^k$, where $|\vec{x}|_1 = x_1 + \cdots + x_n$. Conclude that the Ackermann function is not primitive recursive.