LOGIC EXERCISES – WEEKEND

Exercise 0. Make sure you know how to do the problems from during the week, and think some more about the challenging ones. (Problem 8 from Thursday's homework about spectra is a good example.)

Exercise 1. Let U be a nonprincipal ultrafilter on $\mathbb{N}$, and suppose that $[\mathbb{N}]^2$ is 2-colored. Must there always be a monochromatic set in U?

Exercise 2. Prove that the class of simple groups is not axiomatizable (in the language of groups). Prove that the class of nonabelian simple groups isn't axiomatizable either.

Exercise 3. Let F be a field. Recall from Tuesday's and Wednesday's problem sets how to think of F-vector spaces as first-order structures.

(a) Fix $n \in \mathbb{N}$. Prove that the class of n-dimensional F-vector spaces is axiomatizable.
(b) Prove that the class of infinite-dimensional F-vector spaces is axiomatizable.
(c) Prove that, on the other hand, the class of finite-dimensional F-vector spaces is not axiomatizable. Deduce that the class of infinite-dimensional F-vector spaces is not finitely axiomatizable.

Exercise 4. Take two countable dense linear orders without endpoints. Suppose that each order is *densely painted*, meaning that every point is red or blue, between every pair of distinct blue points there's a red point, and between every pair of distinct red points there's a blue point. Prove that there is an isomorphism from one order to the other that respects the colors of the points.

Exercise 5. (a) Consider the language $\mathcal{L} = \{<\} \cup \{c_n : n \in \mathbb{N}\}$, where $<$ is a binary relation symbol and the c_n are distinct constant symbols. Let $\mathbf{A}$ be the $\mathcal{L}$-structure with underlying set $\mathbb{Q}$, where $<$ is interpreted as the usual ordering on $\mathbb{Q}$ and the constant c_n is interpreted as the natural number n. Show that the theory $T = \mathrm{Th}(\mathbf{A})$ is complete and has exactly 3 countable models up to isomorphism.
(b) Let $n \geq 3$. Find a complete theory with exactly n countable models (up to iso).
(c) $(*)$ Is there a complete theory with exactly 2 countable models (up to iso)?

Exercise 6. $(*)$ Hall's Marriage Theorem is the following statement from finite combinatorics.

> Let $G = (V, E)$ be a bipartite finite graph satisfying "Hall's condition": for all $X \subseteq V$, $|N(X)| \geq |X|$. Then G has a perfect matching, i.e., a set $M \subseteq E$ such that every vertex is incident to exactly one edge in M.

(Here $N(X)$ is the set of neighbors of elements of X.) Prove the following infinitary version from the finite version:

> Suppose that $G = (V, E)$ is a bipartite graph that's locally finite, meaning that every vertex has only finitely many neighbors. If G satisfies Hall's condition — for every $X \subseteq V$, $|N(X)| \geq |X|$ — then G has a perfect matching.

Deduce that in an infinite-dimensional vector space, all bases have the same size.

Exercise 7. The purpose of this exercise is to show that DC is equivalent to LS over ZF, where by LS we mean "every infinite model $\mathbf{A}$ of a countable language $\mathcal{L}$ has a countable elementary submodel." All of following parts of the exercise are to be done in ZF unless otherwise noted (i.e. no form of the Axiom of Choice is to be used).

(a) Recall that DC states: if R is a binary relation on $X \neq \emptyset$ such that for all $x \in X$ there is $y \in X$ with xRy, then there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X so that $x_n R x_{n+1}$ for each $n \in \mathbb{N}$. Prove that DC holds iff every tree T of height ω without terminal nodes has an infinite branch.

(b) Prove (without using any form of Choice, including DC) that if T is a *countable* tree of height ω without terminal nodes, then T has a branch.

(c) Use parts (a) and (b) to show that LS implies DC.

(d) Let $\mathbf{A}$ be a $\mathcal{L}$-structure, $\varphi(x, \vec{y})$ an $\mathcal{L}$-formula ($\vec{y} = (y_1, \ldots, y_n)$), and $A_0 \subseteq A$. We say that a function f is *an A_0-Skolem function for φ* if

$$\mathrm{dom}(f) = \{\vec{a} \in (A_0)^n \colon \mathbf{A} \models \exists x \varphi(x, \vec{a})\},$$

the range of f is a subset of A, and for all $\vec{a}$ in its domain we have $\mathbf{A} \models \varphi(f(\vec{a}), \vec{a})$. Recall DC implies AC_ω, and use this to show that given any structure $\mathbf{A}$ in the countable language $\mathcal{L}$ and countable $A_0 \subseteq A$, we may find a countable collection of A_0-Skolem functions.

(e) Imitate the proof of the (Downwards) Löwenheim-Skolem Theorem given in class to construct a tree of substructures of $\mathbf{A}$, increasing in $\subseteq$ as we ascend the tree. Use DC to select an infinite branch of this tree, and finish as in our proof in class.

Exercise 8.

(a) Prove that the class of 3-colorable graphs is axiomatizable (in the language of graphs). You will probably need to use the previous exercise that says that a graph is 3-colorable iff all of its finite subgraphs are.

(b) ($*$) Prove that the class of non-3-colorable graphs is not axiomatizable, and conclude that 3-colorability is not finitely axiomatizable.

Exercise 9.

(a) Recall (or Google) Kuratowski's theorem about planar graphs. Prove that this planarity criterion fails for infinite graphs, meaning that there is an infinite graph G with no subgraph that is a subdivision of $K_{3,3}$ or K_5, but G is nonetheless not planar. (Find a trivial example and a more interesting example.)

(b) Prove that the class of planar graphs is *axiomatizable for finite structures*, meaning that there is a theory in the language of graphs whose **finite** models are exactly the **finite** planar graphs.

(c) For part (b), would a finite set of axioms do?

(d) Prove that the class of planar graphs is not axiomatizable in the usual sense (i.e., for all structures, not just finite ones).