

## LOGIC EXERCISES – WEEKEND 2

**Exercise 0.** Make sure you know how to do the problems from during the week, and think some more about the challenging ones. You should feel free to revisit the previous weekend’s problems, in particular.

Also, look at problems from last weekend’s set!

**Exercise 1.** Prove that there is a recursive coloring  $[\mathbb{N}]^k \rightarrow \mathbb{N}$  with no *recursive* infinite monochromatic set. (I have in mind an example where  $k = 3$ .)<sup>1</sup>

**Exercise 2** (Do this one!). Let  $x_n$ ,  $n \in \mathbb{N}$ , be countably many members of some  $\mathcal{L}$ -structure  $\mathbf{M}$ . Say that  $(x_n : n \in \mathbb{N})$  is a sequence of **order-indiscernibles** if for all sequences

$$k_1 < k_2 < \cdots < k_m \text{ and } \ell_1 < \ell_2 < \cdots < \ell_m$$

of natural numbers, we have

$$\mathbf{M} \models \phi(x_{k_1}, \dots, x_{k_m}) \leftrightarrow \phi(x_{\ell_1}, \dots, x_{\ell_m})$$

(for all  $\mathcal{L}$ -formulas  $\phi = \phi(v_1, \dots, v_m)$ ).

Let  $T$  be a theory with infinite models. Prove that there is a model  $\mathbf{M} \models T$  and a sequence of order-indiscernibles in  $\mathbf{M}$ .

(Hint: First turn it into a satisfiability problem. You might need Ramsey’s theorem!)

**Exercise 3.**

- (a) Is there a language  $\mathcal{L}$  such that every  $\mathcal{L}$ -structure has only the identity automorphism?
- (b) Is there a language  $\mathcal{L}$  and an  $\mathcal{L}$ -theory  $T$  with infinite models such that every infinite model of  $T$  has no automorphisms other than the identity?

**Exercise 4.** (\*) Let  $G$  be a group. Must there be a language  $\mathcal{L}$  and an  $\mathcal{L}$ -structure  $\mathbf{A}$  such that  $G$  is the group of  $\mathcal{L}$ -automorphisms of  $\mathbf{A}$ ?

**Exercise 5.** Use the compactness theorem (i.e., don’t use a stronger form of AC) to prove that every partial order extends to a linear order.

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<sup>1</sup>“What does it mean for a function  $f: [\mathbb{N}]^k \rightarrow \mathbb{N}$  to be recursive?” you ask. One sensible way to interpret this is that the function  $f': \mathbb{N}^k \rightarrow \mathbb{N}$  is recursive, where  $f'$  is given by  $f'(a_1, \dots, a_k) = f(\{a_1, \dots, a_k\})$  when  $a_1, \dots, a_k$  are distinct and  $f'(a_1, \dots, a_k) = 0$  otherwise.

**Exercise 6.** Let  $<_1$  and  $<_2$  be recursive dense linear orderings of  $\mathbb{N}$  without endpoints. Must there be a recursive isomorphism between them?

**Exercise 7** (Kleene's Second Recursion Theorem). For each partial recursive function  $f = f(z, \vec{x})$ , there is a number  $z^*$  such that for all  $\vec{x}$ ,

$$\varphi_{z^*}(x) = \{z^*\}(\vec{x}) = f(z^*, \vec{x}).$$

(Prove this.)

We say that  $A \subseteq \mathbb{N}$  is an *index set* if  $e \in A$  and  $\varphi_e = \varphi_d$  implies  $d \in A$ .

**Exercise 8** (Rice's Theorem). If  $E \subseteq \mathbb{N}$  is a recursive index set, then  $E = \emptyset$  or  $E = \mathbb{N}$ .

**Exercise 9.** If  $T$  has a recursively enumerable set of axioms, then it also has a decidable set of axioms (in the same language). (What I mean here is that the set of Gödel codes of the axioms is a recursive (or r.e.) set of integers.)

**Exercise 10.** Prove that  $K$  is r.e.