# FORCING EXERCISES – DAY 10/WEEKEND 2

**Exercise 0.** Make sure you know how to do the problems from the past few days, especially the questions about forcing. (Seriously. Do them.)

**Exercise 1.** Let $\mathbb{P} \in M$ be a poset and let $p \in \mathbb{P}$. We say that a set $D \subseteq \mathbb{P}$ is **dense below** $p$ if every $q \leq p$ has an extension $r \leq q$ with $r \in D$.

(a) Show that if $G$ is $\mathbb{P}$-generic over $M$ and $D$ is dense below $p \in G$, then $G \cap D \neq \emptyset$.

(b) Let $\phi$ be a sentence in the forcing language. Suppose that the set $D = \{q \in \mathbb{P} : q \Vdash \phi\}$ is dense below $p$. Show that $p \in D$, i.e., $p \Vdash \phi$.

**Exercise 2.** Suppose that $G$ is $\mathbb{P}$-generic over $M$, where $\mathbb{P}$ is countable (in $M$). Show that if $A$ is an uncountable set of ordinals in $M[G]$, then there is $B \subseteq A$, also uncountable (in $M[G]$), with $B \in M$.

**Exercise 3.** An **automorphism** of a poset $\mathbb{P}$ is a bijection $i\colon \mathbb{P} \to \mathbb{P}$ such that $p \leq q$ iff $i(p) \leq i(q)$. An automorphism $i$ of $\mathbb{P}$ in $M$ induces a permutation of the set $M^{\mathbb{P}}$ of names by the following recursive definition for $\mathbb{P}$-names $\tau$:

$$i(\tau) = \{\langle i(\sigma), i(p)\rangle : \langle \sigma, p\rangle \in \tau\}.$$

(We call the permutation of names $i$ too.)

(a) Show that if $i\colon \mathbb{P} \to \mathbb{P}$ is an automorphism, then $i(\check{x}) = \check{x}$ for every $x \in M$.

(b) Let $\tau \in M^{\mathbb{P}}$. Prove that $p \Vdash \phi[\tau]$ iff $i(p) \Vdash \phi[i(\tau)]$.

**Exercise 4.** The Cohen poset $\mathbb{C}$ consists of finite partial functions $\omega \to \omega$, ordered by reverse inclusion. A **Cohen real over** $M$ is a function $c\colon \omega \to \omega$ such that the corresponding filter $G = \{p \in \mathbb{C} : p \subseteq c\}$ is $\mathbb{C}$-generic over $M$.

(a) Show that if $G$ is a $\mathbb{C}$-generic filter over $M$ then the function $g = \bigcup G$ is a Cohen real over $M$. (So $\mathbb{C}$-generic filters and Cohen reals are essentially the same thing.)

(b) Suppose that $c$ is a Cohen real over $M$. Show that the function $n \mapsto c(2n)$ is also a Cohen real over $M$.

(c) More generally, suppose that $A \subseteq \omega$ is any infinite subset *belonging* to $M$. Let $e_A\colon \omega \to \omega$ enumerate $A$ in increasing order. Show that the function $n \mapsto c(e_A(n))$ is a Cohen real over $M$.

(d) Give an example of an infinite $A \subseteq \omega$ belonging to $M[c]$ such that the function $n \mapsto c(e_A(n))$ is not a Cohen real over $M$.

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**Exercise 5.** Suppose that $c\colon \omega \to \omega$ is a Cohen real over $M$. (See Exercise 4.) Prove that $c$ is not dominated by any function $\omega \to \omega$ that belongs to $M$.

**Exercise 6.** Recall that $x$ **splits** $y$ if $|y \cap x| = |y \setminus x| = \aleph_0$. Suppose that $G$ is $\mathrm{Fn}(\omega, 2)$-generic over $M$ and put $g = \{n < \omega : (\exists p \in G)\, p(n) = 1\}$.

(a) Prove that $g$ has infinitely many even members and infinitely many odd members.
(b) Prove that $g$ is split by any infinite, coinfinite subset $y \subseteq \omega$ that belongs to $M$.
(c) Prove that there are infinitely many even numbers in $g$ and infinitely many even numbers not in $g$.
(d) Prove that $g$ splits every $y \in [\omega]^\omega \cap M$.

**Exercise 7.** Suppose that $\mathbb{P}$ is a separative poset in $M$. Show that the set

$$\{\tau \in M^{\mathbb{P}} : \tau[G] = \emptyset \text{ for every } M\text{-generic filter } G\}$$

is an element of $M$, but the set

$$\{\tau \in M^{\mathbb{P}} : \tau[G] = \emptyset \text{ for some } M\text{-generic filter } G\}$$

is not. (Hint: Think about $\mathbb{P}$-rank.)

**Exercise 8** (FERMAT'S LAST THEOREM MOD P, OMGOMG!)**.**

(a) Fix $r \in \mathbb{N}$. Prove that there is $n \in \mathbb{N}$ such that whenever $n = \{0, \dots, n-1\}$ is $r$-colored (the elements, not pairs, are colored in $r$ colors), there is a monochromatic solution to $x + y = z$, i.e., a pair of elements $x, y < n$ such that $\{x, y, x + y\}$ is a monochromatic set. (Hint: You needn't do it from scratch.)
(b) Prove, using part (a) if you want, that for every $n$ the equation $x^n + y^n = z^n$ has a nontrivial solution mod $p$ for all sufficiently large $p$.

**Exercise 9** (Weekend 1, #1)**.** Prove that $\mathfrak{s} \leq \mathfrak{d}$.

**Exercise 10.** If $\mathbb{P}$ is a poset, then $\mathrm{cc}(\mathbb{P})$ is the least cardinal $\kappa$ such that $\mathbb{P}$ has no antichains of size $\kappa$. (So $\mathbb{P}$ has the ccc iff $\mathrm{cc}(\mathbb{P}) \leq \aleph_1$.)

(a) Suppose that $\mathrm{cc}(\mathbb{P}) > n$ for every $n < \omega$. Show that $\mathrm{cc}(\mathbb{P}) > \omega$, i.e., that $\mathbb{P}$ has an infinite antichain.
(b) $(*)$ Show that $\mathrm{cc}(\mathbb{P})$ must be a regular cardinal.

Email Zach if you have questions or need a hint.