FORCING EXERCISES – DAY 11

Exercise 0. Make sure you know how to do the first 7 problems from the weekend. (Seriously. Do them.)

Exercise 1. Let $\mathbb{P} \in M$ be a separative partial order, and let $\dot{G}$ be the canonical name for the generic filter, as defined in your notes. Show for $q, p \in \mathbb{P}$ that $q \Vdash \check{p} \in \dot{G}$ iff $q \leq p$.

Exercise 2. Let $\mathbb{P} \in M$ be a poset, and let $\mathbb{Q}$ be a dense subset of $\mathbb{P}$, viewed as a subposet. Prove the following.

(a) If G is $\mathbb{P}$-generic, then $H = G \cap \mathbb{Q}$ is a $\mathbb{Q}$-generic filter. Furthermore, $G = \{p \in \mathbb{P} : (\exists q \in H)\, q \leq p\}$.
(b) If H is $\mathbb{Q}$-generic, then the set $G = \{p \in \mathbb{P} : (\exists q \in H)\, q \leq p\}$ is $\mathbb{P}$-generic. Furthermore, $H = G \cap \mathbb{Q}$.
(c) If G and H are taken as in either of the items above, then $M[G] = M[H]$.
(d) Any $\mathbb{Q}$-name is a $\mathbb{P}$-name, and if $\tau \in M^{\mathbb{Q}}$ then $1_{\mathbb{P}} \Vdash_{\mathbb{P}} \phi[\tau]$ iff $1_{\mathbb{Q}} \Vdash_{\mathbb{Q}} \phi[\tau]$.
(e) For any $\tau \in M^{\mathbb{P}}$ there is a $\sigma \in M^{\mathbb{Q}}$ such that $1_{\mathbb{P}} \Vdash_{\mathbb{P}} \tau = \sigma$.

Exercise 3. Let $\mathbb{P}$ be the poset $[\omega]^\omega$, ordered by almost-inclusion: $p \leq q$ iff $p \subseteq^* q$.

(a) What are antichains in $\mathbb{P}$? What is $\mathrm{cc}(\mathbb{P})$? (See Weekend 2 for the definition.)
(b) Prove that $\mathbb{P}$ is countably closed and conclude that $\mathbb{P}$ **doesn't add reals**, meaning that $M \cap [\omega]^\omega = M[G] \cap [\omega]^\omega$ whenever G is $\mathbb{P}$-generic over M.
(c) Prove that in $M[G]$ the $\mathbb{P}$-generic G is a nonprincipal ultrafilter on ω.
(d) Prove that, in fact, G is a Ramsey ultrafilter.

Exercise 4.

(a) Suppose that $A \subseteq \mathbb{P}$ is a maximal antichain in M and that τ_p is a $\mathbb{P}$-name for each $p \in A$. Let $\pi = \{\langle \tau_p, p\rangle : p \in A\}$ and observe that π is a $\mathbb{P}$-name. What is the cardinality of $\pi[G]$?
(b) Show that if $p \Vdash (\exists x)\phi(x)$, then there is a name $\sigma \in M^{\mathbb{P}}$ such that $p \Vdash \phi[\sigma]$.

Email Zach if you have questions or need a hint.

Date: 25 July 2016.