FORCING EXERCISES – DAY 12

Exercise 0. Make sure you know how to do the problems from yesterday.

Exercise 1. A poset $\mathbb{P}$ is **kinda homogeneous** if for any $p, q \in \mathbb{P}$ there is an automorphism $i\colon \mathbb{P} \to \mathbb{P}$ such that $i(p)$ and q are compatible.

(a) Show that if I is an infinite set and J is any nonempty set then the poset

$$\mathrm{Fn}(I, J) = \{p : p \text{ is a function and } \mathrm{dom}(p) \subseteq I \text{ is finite and } \mathrm{ran}(p) \subseteq J\},$$

ordered by reverse inclusion, is kinda homogeneous.

(b) Suppose that $\mathbb{P}$ is kinda homogeneous and G is $\mathbb{P}$-generic. Let $x \in M$. Show that if $M[G] \models \phi[x]$, then in fact $1_{\mathbb{P}} \Vdash \phi(\check{x})$.

(c) Conclude that if $\mathbb{P}$ is kinda homogeneous then for any $\mathbb{P}$-generic filters G and H, the generic extensions $M[G]$ and $M[H]$ have the same first-order theory (that is, they're elementarily equivalent).

Definition. If $M \subseteq N$ are countable transitive models of ZFC, then we say $f \in \omega^\omega \cap N$ is a **dominating real over** N if f dominates every function $\omega \to \omega$ that belongs to M.

Exercise 2.

(a) Prove that the Cohen-real forcing (from Weekend 2 #4—conditions are finite partial functions $\omega \to \omega$) doesn't add any dominating reals over the ground model. (Caution: there's more to this than proving that the generic real is not a dominating real, but that might be a good place to start.)

(b) Prove that if $A \in M$ is a mad family (in M) and $d \in N$ is a dominating real, then A is not mad (i.e., not maximal) in N. (Hint: look at the proof of $\mathfrak{b} \leq \mathfrak{a}$.)

Exercise 3 (Weekend 2, #7)**.** Suppose that $\mathbb{P}$ is a separative poset in M. Show that the set

$$\{\tau \in M^{\mathbb{P}} : \tau[G] = \emptyset \text{ for every } M\text{-generic filter } G\}$$

is an element of M, but the set

$$\{\tau \in M^{\mathbb{P}} : \tau[G] = \emptyset \text{ for some } M\text{-generic filter } G\}$$

is not. (Hint: Think about $\mathbb{P}$-rank.)

Exercise 4. $(*)$ Assume CH. Prove that there is an infinite mad family $A \subseteq [\omega]^\omega$ that remains mad in $M[G]$ for any G that is Cohen-generic over M. (The poset here is the poset of finite partial functions $\omega \to \omega$ again.)

Email Zach if you have questions or need a hint.

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