FORCING EXERCISES (THE LAST!) – DAY 13

Exercise 0. Make sure you know how to do the problems from yesterday.

Exercise 1. A set x is **ordinal-definable** if there are a formula ϕ in the language of set theory and finitely many ordinals $\alpha_0, \dots, \alpha_n$ such that $x = \{y : \phi(y, \alpha_0, \dots, \alpha_n)\}$.

(a) Reflect on why it isn't obvious that the class of ordinal-definable sets is a definable class.

(b) Show that, nevertheless, a set is ordinal-definable iff there are a formula ϕ and ordinals β and $\alpha_0, \dots, \alpha_n < \beta$ such that $x = \{y \in V_\beta : V_\beta \models \phi(y, \alpha_0, \dots, \alpha_n)\}$. (So it is a definable class, after all.)

Exercise 2. Show that if κ is a strongly inaccessible cardinal in M and $\mathbb{P}$ is a poset of cardinality $< \kappa$ (in M), then κ remains strongly inaccessible in any generic extension $M[G]$ by $\mathbb{P}$.

Exercise 3. Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are posets in M. Prove that a filter $F \subseteq \mathbb{P} \times \mathbb{Q}$ is $\mathbb{P} \times \mathbb{Q}$-generic over M iff there are filters G, H such that $F = G \times H$, G is $\mathbb{P}$-generic over M, and H is $\mathbb{Q}$-generic over $M[G]$. (In this case, $M[F] = M[G][H]$.)

Exercise 4. Suppose that M is a transitive model of ZFC and that

$$M \models \text{“}X \text{ and } Y \text{ are closed subsets of the unit interval } [0,1]\text{”}.$$

(You may assume if you wish that $M \cap [0,1]$ is countable, though that shouldn't be necessary.) Notice that the sets M thinks are closed are unlikely to be closed. Prove that

(a) $\mu^M(X) = \mu(\overline{X})$.

(b) if $X \cap Y = \emptyset$, then $\overline{X} \cap \overline{Y} = \emptyset$.

(c) $\overline{X \cap Y} = \overline{X} \cap \overline{Y}$.

(Here $\overline{X}$ is the closure of X in the true unit interval.)

Email Zach if you have questions or need a hint.

Date: 27 July 2016.