FORCING EXERCISES – DAY 2

Exercise 0. Make sure you know how to do the problems from yesterday.

Exercise 1. ~~Let κ be an infinite cardinal. Show that κ^+ is a regular cardinal.~~

Exercise 1.

(a) Show that $\mathsf{MA}(\kappa)$ is equivalent to the same statement if we replace 'dense' by 'dense open'.

(b) Show that $\mathsf{MA}(\kappa)$ is equivalent to the same statement if we replace 'dense sets' by 'maximal antichains'. (That is, $\mathsf{MA}(\kappa)$ is equivalent to the statement: for every ccc poset $\mathbb{P}$ and every collection C of $\leq \kappa$ maximal antichains of $\mathbb{P}$, there is a filter that meets every maximal antichain in C.)

Exercise 2. Show that the bounding number $\mathfrak{b}$ is a regular cardinal.

Definition. For infinite sets $x, y \subseteq \omega$, we say that x **splits** y if $y \cap x$ and $y \setminus x$ are both infinite. (Draw a picture.) A family $F \subseteq \mathscr{P}(\omega)$ is a **splitting family** if each infinite $y \subseteq \omega$ is split by at least one $x \in F$. The **splitting number** $\mathfrak{s}$ is the least cardinality of a splitting family.

Exercise 3. Show that $\aleph_0 < \mathfrak{s} \leq 2^{\aleph_0}$.

Definition. Let $x, y \subseteq \omega$. As usual, we write $x \subseteq^* y$ to mean that $x \setminus y$ is finite. A sequence $\langle x_\alpha : \alpha < \kappa \rangle$ of distinct infinite subsets of ω is a **tower** if $x_\beta \subseteq^* x_\alpha$ whether $\alpha < \beta$. The **tower number** $\mathfrak{t}$ is the minimal length of a maximal tower. (A maximal tower is one for which no further set is almost-contained in every member of the tower.)

Exercise 4.

(a) Show that $\aleph_0 < \mathfrak{t} \leq 2^{\aleph_0}$.

(b) Show that $\mathfrak{t}$ is a regular cardinal.

(c) Show that $\mathfrak{t} \leq \mathfrak{s}$.

Exercise 5. Show that there is a mad family of cardinality $2^{\aleph_0}$.

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