FORCING EXERCISES – DAY 3

Exercise 0. Make sure you know how to do the problems from yesterday.

Exercise 1. Prove the following:

(a) If A is a mad family, then its downward closure

$$\{x \in [\omega]^\omega : (\exists y \in A)\, x \subseteq^* y\}$$

is a dense open family.

(b) Every dense open family includes a mad family.

(For the definition of the tower number $\mathfrak{t}$, see yesterday's problem set.)

Exercise 2. Suppose that κ is a cardinal and that $\aleph_0 \leq \kappa < \mathfrak{t}$. Prove that $2^\kappa = 2^{\aleph_0}$.

Deduce that $\mathfrak{t} \leq \mathrm{cf}(2^{\aleph_0})$.

Exercise 3. Prove that MA implies $\mathfrak{t} = 2^{\aleph_0}$. Deduce (again) that MA implies that $2^{\aleph_0}$ is regular.

Definition. The **distributivity number** or **s$\mathfrak{h}$attering number** $\mathfrak{h}$ is the smallest number of dense open families $\subseteq [\omega]^\omega$ with empty intersection.

In prep exercise #19, you proved that $\mathfrak{h}$ is uncountable. (Make sure you know how to do that one!)

Exercise 4.

(a) Prove that $\mathfrak{h}$ is a regular cardinal.

(b) Prove that $\mathfrak{t} \leq \mathfrak{h}$.

Exercise 5. Let $\mathfrak{z}$ be the least size of a family $F \subseteq [\omega]^\omega$ such that every dense open family $D \subseteq [\omega]^\omega$ meets F.

(a) Prove that $\mathfrak{z} = 2^{\aleph_0}$.

(b) Prove that $\mathfrak{h} \leq \mathrm{cf}(2^{\aleph_0})$. (Hint: I put this here for a reason.)

Date: 13 July 2016.