FORCING EXERCISES – DAY 4

Exercise 0. Make sure you know how to do the problems from yesterday.

Exercise 1. If you haven't seen measure theory before, prove any unproven facts about Lebesgue measure from lecture.

Definition. Let $\mathbb{P}$ be a poset.

(a) A subset $X \subseteq \mathbb{P}$ is **centered** if any finitely many members of X have a common extension (in $\mathbb{P}$). Say that $\mathbb{P}$ is **σ-centered** if it can be partitioned into countably many sets, each of which is centered.
(b) $\mathbb{P}$ has the **$\aleph_1$-Knaster property** if every uncountable subset $Y \subseteq \mathbb{P}$ has an uncountable subset $X \subseteq Y$ with the property that any two conditions in X are compatible.

Exercise 2. Show that

$$\sigma\text{-centered} \implies \aleph_1\text{-Knaster} \implies \text{ccc}$$

for posets.

Exercise 3. Suppose that $\{A_n : n < \omega\} \subseteq \mathcal{L}$ and for all n, $A_n \subseteq A_{n+1}$. Show that $\mu(\bigcup_{n<\omega} A_n) = \lim_{n<\omega} \mu(A_n)$.

Exercise 4.

(a) Show that every open subset of $\mathbb{R}$ can be written as the union of open intervals with rational endpoints.
(b) Show that there are exactly $2^{\aleph_0}$ open subsets of $\mathbb{R}$.
(c) Show that there are exactly $2^{\aleph_0}$ Borel subsets of $\mathbb{R}$.

Exercise 5. Show that there are Lebesgue-measure-zero sets that are not Borel.

Exercise 6. We define a poset $\mathbb{B}$ as follows. Conditions are subsets of the interval $(0, 1)$ that have positive Lebesgue measure. They are ordered by $p \leq q$ iff $p \subseteq q$.

(a) Give a characterization of $p \perp q$ for $p, q \in \mathbb{B}$.
(b) Show that if $\{p_n : n < \omega\}$ is an antichain, then $\mu(\bigcup_{n<\omega} p_n) = \sum_{n<\omega} \mu(p_n)$.
(c) Show that $\mathbb{B}$ has the ccc. Is it σ-centered?

Exercise 7. Recall from yesterday's set the definition of the distributivity number $\mathfrak{h}$.

(a) Prove that $\mathfrak{h} \leq \mathfrak{s}$.
(b) Prove that $\mathfrak{h} \leq \mathfrak{b}$.

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