FORCING EXERCISES – DAY 5

Exercise 1. A family $F \subseteq \omega^\omega$ is called **super-unbounded** if, for every function $g \in \omega^\omega$, the set $\{f \in F : f \leq^* g\}$ has cardinality $< |F|$. Prove that there is always a super-unbounded family of size $\mathfrak{b}$.

Exercise 2. Let $A \subseteq [\omega]^\omega$ be a mad family. Prove that there is no uniform (finite) bound on the sizes of intersections of members of A. That is, prove that for every $m < \omega$ there are $n > m$ and distinct $x, y \in A$ such that $|x \cap y| \geq n$.

Exercise 3.
(a) Show that $\mathbb{Q}$ is not a G_δ subset of $\mathbb{R}$.
(b) Show that there is a dense G_δ set of measure zero.

Exercise 4. Suppose that $X \subseteq \mathbb{R}$ is an uncountable set of cardinality $< 2^{\aleph_0}$. (So X witnesses the failure of CH.) Prove that X is either nonmeasurable or has measure zero.

Exercise 5. Prove that there is no closed counterexample to CH. That is, prove that if $X \subseteq \mathbb{R}$ is a closed set, then X is either countable or has cardinality $2^{\aleph_0}$. (This fact is also true for all Borel subsets of $\mathbb{R}$, but I won't make you prove that.)

Exercise 6. Prove Ramsey's theorem for arbitrary (finite) arity of colorings and for any (finite) number of colors. That is, prove that for any $k, r \in \mathbb{N}$, if $[\mathbb{N}]^k$ is r-colored, then there is an infinite monochromatic set. (Hint: You don't need to reprove the theorem.)

Exercise 7. (a) Prove that $\aleph_0 < \mathfrak{par} \leq 2^{\aleph_0}$.
(b) Prove that $\mathfrak{d} \leq \mathfrak{hom}$.

Definition. Recall that a set X generates an ultrafilter on ω if (X generates a filter and) the smallest filter including X is an ultrafilter. The **ultrafilter number** $\mathfrak{u}$ is the smallest size of a set that generates a nonprincipal ultrafilter on ω.

Exercise 8. (a) Give an example of two non-ultra filters F and G on ω whose union $F \cup G$ generates a nonprincipal ultrafilter.
(b) Show that (on the other hand) if $F_0 \subseteq F_1 \subseteq \cdots$ is a countable increasing sequence of filters on ω and each F_n is *not* ultra, then the union $\bigcup_{n<\omega} F_n$ does not generate an ultrafilter. Conclude that $\mathfrak{u}$ cannot have countable cofinality. (E.g. $\mathfrak{u} \neq \aleph_\omega$.)
(c) Prove that $\aleph_0 < \mathfrak{u} \leq 2^{\aleph_0}$.
(d) Prove that $\mathfrak{u} = 2^{\aleph_0}$ under MA. (Don't get carried away!)

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