FORCING EXERCISES – DAY 6

Exercise 1. Express the axioms for a dense linear order without endpoints as sentences in the language of a single binary relation $<$.

Exercise 2. The language for set theory consists of a single binary relation symbol, $\in$. Consider the following as $\{\in\}$-structures. Which axioms of ZFC does each satisfy?
(a) $(\mathbb{Z}, <)$
(b) $(\omega, \in)$
(c) $(\omega_1, \in)$
(d) $(V_\omega, \in)$

Exercise 3. Prove that if ZFC is consistent, then there is an illfounded model of ZFC, i.e., a model $(M, E) \models \mathsf{ZFC}$ such that $E = \in^M$ is an illfounded binary relation on M.

Exercise 4. Prove that 2^ω is a compact metric space. Deduce that $2^{[\omega]^2}$ is compact. Use this fact to derive the following finitary version of Ramsey's theorem from the infinitary one:

Theorem. For every $k < \omega$ there is an $n < \omega$ such that every coloring $[n]^2 \to 2$ has a monochromatic set of size k.[1]

Exercise 5. Recall that a filter F on ω is a Ramsey filter if every 2-coloring of $[\omega]^2$ has a monochromatic set in F. Prove that a Ramsey filter must be a nonprincipal ultrafilter.

Exercise 6. If $A \subseteq [\omega]^\omega$ is a countably infinite almost-disjoint family and there is a mad family of size κ, then A extends to a mad family of size κ. Prove this.

Definition. A family $I \subseteq [\omega]^\omega$ is called **independent** if the intersection of any finitely many members of I with the complements of any finitely many other members of I is infinite.

Exercise 7. Prove that there is an independent family of size $2^{\aleph_0}$. Prove that there are $2^{2^{\aleph_0}}$ ultrafilters on ω (using the independent family or otherwise).

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[1]NB. The least n suitable for $k = 5$ is not known! It's known to be between 43 and 49, though.