FORCING EXERCISES – DAY 7

Exercise 0. Make sure you know how to do the problems from yesterday.

Exercise 1. Suppose that M is a transitive set and $\alpha \in M$. Then

$$\langle M, \in \rangle \models \text{“}\alpha \text{ is a cardinal”},$$

if α is truly a cardinal. (NB. The converse is false.)

Exercise 2. Let X be a countable elementary substructure of some H_θ, where θ is regular and uncountable. (Note that X is not transitive!)

(a) Suppose that $A \in X$ and $H_\theta \models$ “A is countable”. Show that then $X \models$ “A is countable” and $A \subseteq X$.
(b) Show that $X \cap \omega_1 \in \omega_1$, i.e., $X \cap \omega_1$ is a countable ordinal.
(c) Define a countable set of ordinals that is not a member of X.
(d) Show that $\omega_1 \in X$ if $\theta > \omega_1$.
(e) Describe a subset of ω that is not a member of X.

Exercise 3. A cardinal κ is called **inaccessible** if it is a regular limit cardinal; it's called **strongly inaccessible** if in addition $2^\lambda < \kappa$ for every $\lambda < \kappa$.

(a) If κ is strongly inaccessible, then $V_\kappa \models \mathsf{ZFC}$. Sketch a proof.
(b) If $V_\kappa \models \mathsf{ZFC}$, does it follow that κ is an inaccessible cardinal?

Exercise 4. A theory T is **finitely axiomatizable** if there is a finite set $A \subseteq T$ such that $A \models T$; that is, if $A \models \sigma$ for every sentence $\sigma \in T$. Show that ZFC is not finitely axiomatizable.

Exercise 5. Suppose $\mathfrak{a} < 2^{\aleph_0}$. Prove that there is an open dense set $D \subseteq [\omega]^\omega$ that doesn't include a mad family of size $\mathfrak{a}$ (though it does include a mad family).

Exercise 6. Instead of finding monochromatic sets, you might try looking for polychromatic ones. Suppose that $[\omega]^2$ is colored (using infinitely many colors) in a way that is k-**restricted**, meaning that each color is used at most k times. Prove that there is an infinite **fully polychromatic** set X, i.e., a set X such that on pairs of elements of X each color is used at most once.

Date: 19 July 2016.