FORCING EXERCISES – DAY 8

Exercise 0. Make sure you know how to do the problems from yesterday.

Exercise 1. Prove that all the assertions in the proposition from lecture can be expressed by Δ_0-formulas.

Definition. A formula in the language of set theory is a Σ_1**-formula** if it of the form $\exists u\, \psi(u, v)$, where ψ is a Δ_0-formula. A Π_1**-formula** is one of the form $\forall u\, \psi(u, v)$, where ψ is a Δ_0-formula. We say that a formula $\phi = \phi(v)$ is a Δ_1**-formula** if it's ZFC-provably-equivalent to both a Σ_1-formula and a Π_1-formula. What we mean by this is that there is a Σ_1 formula θ and a Π_1-formula π such that

$$\mathsf{ZFC} \vdash \forall v\, (\phi(v) \leftrightarrow \theta(v)) \quad \text{and} \quad \mathsf{ZFC} \vdash \forall v\, (\phi(v) \leftrightarrow \pi(v)).$$

Exercise 2. Show that Δ_1-formulas are absolute for transitive models of ZFC.

Exercise 3. Show that "R is a wellfounded relation" is absolute for transitive models of ZFC. (Hint: To show that it's Σ_1, construct a ranking function.)

Exercise 4. A set $E \subseteq \mathbb{P}$ is **predense** if every $p \in \mathbb{P}$ is compatible with at least one $q \in E$. Show that the following are equivalent conditions on a filter $G \subseteq \mathbb{P}$.

(a) G is $\mathbb{P}$-generic over M.
(b) G meets every maximal antichain $A \in M$.
(c) G meets every predense set $E \in M$.

(Compare with #1 on Day 2's problem set.)

Exercise 5. You might wonder whether Ramsey's theorem holds for colorings of the infinite subsets of $\mathbb{N}$. Prove that it doesn't. That is, show that there is a coloring $\chi\colon [\omega]^\omega \to 2$ such that no infinite $X \subseteq \omega$ is monochromatic. (An infinite set $X \subseteq \omega$ is monochromatic if $\chi \restriction [X]^\omega$ is constant.) Notice anything about your proof?

Exercise 6. Prove that $\mathfrak{u} \geq \mathfrak{b}$.

Date: 20 July 2016.