# FORCING EXERCISES – DAY 9

**Exercise 0.** Make sure you know how to do the problems from yesterday.

**Exercise 1.** Suppose that $p \Vdash \phi$ and $\mathsf{ZFC} \vdash (\phi \to \psi)$. Prove that $p \Vdash \psi$. (You may assume that $M[G] \models \mathsf{ZFC}$.)

**Exercise 2.** Prove the following.

(a) For any formulas $\phi = \phi(v)$ and $\psi = \psi(v)$, names $\tau, \sigma \in M^{\mathbb{P}}$, and condition $p \in \mathbb{P}$,
$$p \Vdash \phi[\tau] \wedge \psi[\sigma] \text{ iff both } p \Vdash \phi[\tau] \text{ and } p \Vdash \psi[\sigma].$$

(b) For any formula $\phi = \phi(v)$ and name $\tau \in M^{\mathbb{P}}$, $p \Vdash \neg\phi[\tau]$ iff there is no $q \leq p$ such that $q \Vdash \phi[\tau]$.

**Exercise 3.** Let $\mathbb{P}$ be the poset of functions $p \colon n \to \omega$ for all $n \in \omega$, ordered by reverse inclusion: $p \leq q$ iff $p \supseteq q$. For $n \in \omega$ let $D_n = \{p : n \in \mathrm{dom}(p)\}$. By $\mathsf{MA}(\omega)$ we know that there is a filter $G \subseteq \mathbb{P}$ meeting each dense set $D_n$, and $g = \bigcup G$ is a (total) function $\omega \to \omega$. For each of the following, give a $\mathbb{P}$-name $\tau$ that satisfies the given conditions for all filters $G$ that meet each $D_n$.

(a) $\tau[G] = 1$ if $g(0) = 0$ and $\tau[G] = 0$ otherwise.
(b) $\tau[G] = 1$ if $4 \in \mathrm{ran}(g)$ and $\tau[G] = 0$ otherwise.
(c) $\tau[G] = \omega$ iff $g$ is surjective.
(d) $\tau[G] = g(0)$.
(e) $\tau[G] = \{g(0)\}$.
(f) $\tau[G] = g^{-1}\{0\}$.

**Exercise 4** (Same setup as previous problem)**.** Also show that for no name $\tau$ do we have $\tau[G] = 1$ iff $g$ is surjective.

**Exercise 5.** (a) Let $\sigma, \tau \in M^{\mathbb{P}}$ be $\mathbb{P}$-names. Construct a name $\pi \in M^{\mathbb{P}}$ so that $\pi[G] = \sigma[G] \cup \tau[G]$, no matter what the filter $G$ is.

(b) Let $\sigma \in M^{\mathbb{P}}$. Construct a name $\pi \in M^{\mathbb{P}}$ so that $\pi[G] = \bigcup \sigma[G]$, no matter what the filter $G$ is.

**Exercise 6.** Let $\tau \in M^{\mathbb{P}}$ with $\mathrm{dom}(\tau) \subseteq \{\check{n} : n < \omega\}$. Construct a name $\sigma \in M^{\mathbb{P}}$ so that $\sigma[G] = \omega \setminus \tau[G]$, no matter what the filter $G$ is.[1]

**Exercise 7.** Prove that the poset $\mathbb{B}$ from Day 4, #6, is not $\sigma$-centered.

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[1] Assume that $G$ meets each of the countably many dense sets $D_n = \{p : p \Vdash \check{n} \in \tau \text{ or } p \Vdash \check{n} \notin \tau\}$.