FORCING SUMMER SCHOOL LECTURE NOTES

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Abstract. These are the lecture notes for the forcing class in the 2014 UCLA Logic Summer School. They are a revision of the 2013 Summer School notes. The most substantial differences are a new proof of the forcing theorems in section 10.3 and a final section on the consistency of the failure of the axiom of choice. Both write-ups are by Sherwood Hachtman; the former is based on the author's notes from a course given by Richard Laver.

These notes would not be what they are without a previous set of lectures by Justin Palumbo. The introduction to the forcing language and write-ups of basic results in sections 10.1 and 10.2 are his, and many of the other lectures follow his notes.

The Axioms of ZFC, Zermelo-Fraenkel Set Theory with Choice

- *Extensionality*: Two sets are equal if and only if they have the same elements.
- *Pairing*: If a and b are sets, then so is the pair $\{a, b\}$.
- *Comprehension Scheme*: For any definable property $\phi(u)$ and set z, the collection of $x \in z$ such that $\phi(x)$ holds, is a set.
- *Union*: If $\{A_i\}_{i\in I}$ is a set, then so is its union, $\bigcup_{i\in I} A_i$.
- *Power Set*: If X is a set, then so is $\mathcal{P}(X)$, the collection of subsets of X.
- *Infinity*: There is an infinite set.
- *Replacement Scheme*: For any definable property $\phi(u, v)$, if ϕ defines a function on a set a, then the pointwise image of a by ϕ is a set.
- *Foundation*: The membership relation, $\in$, is well-founded; i.e., every nonempty set contains a $\in$-minimal element.
- *Choice*: If $\{A_i\}_{i\in I}$ is a collection of nonempty sets, then there exists a choice function f with domain I, so that $f(i) \in A_i$ for all $i \in I$.

Foundation is equivalent to the statement that every set belongs to some V_α; Choice is equivalent to the statement that every set can be well-ordered (Zermelo's Theorem). ZFC without the Axiom of Choice is called ZF.

§1. The Continuum Problem. The most fundamental notion in set theory is that of well-foundedness.

Definition 1.1. A binary relation R on a set A is **well-founded** if every nonempty subset $B \subseteq A$ has a minimal element, that is, an element c such that for all $b \in B$, $b \mathrel{R} c$ fails.

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Definition 1.2. A linear order $<$ on a set W is a **well-ordering** if it is well-founded.

Remark 1.3. We collect some remarks:

- In a well-order if c is a minimal element, then $c \leq b$ for all b. So 'minimal' in this case means 'least'.
- Every finite linear order is a well-order.
- The set of natural numbers is a well-ordered set, but the set of integers is not.
- The Axiom of Choice is equivalent to the statement 'Every set can be well-ordered'.

We will now characterize all well-orderings in terms of ordinals. Here are a few definitions.

Definition 1.4. A set z is **transitive** if for all $y \in z$ and $x \in y$, $x \in z$.

Definition 1.5. A set α is an **ordinal** if it's transitive and well-ordered by $\in$.

Proposition 1.6. *We have the following easy facts:*

1. $\varnothing$ *is an ordinal.*
2. *If* α *is a ordinal, then the least ordinal greater than* α *is* $\alpha \cup \{\alpha\}$. *We call this ordinal* $\alpha + 1$.
3. *If* $\{\alpha_i \mid i \in I\}$ *is a collection of ordinals, then* $\bigcup_{i \in I} \alpha_i$ *is an ordinal.*
4. *We write* On *for the class of ordinals.* On *is well-ordered by* $\in$.

Ordinals from the bottom up:

- $0 = \varnothing$
- $1 = \{\varnothing\}$
- $2 = \{\varnothing, \{\varnothing\}\}$
- $3 = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$

  $\vdots$
- $\omega = \{0, 1, 2, 3...\}$
- $\omega + 1 = \{0, 1, 2, 3, ..., \omega\}$

The next proposition captures why ordinals are interesting.

Proposition 1.7. *Every well-ordering is isomorphic to a unique ordinal: For every well-ordering* $(W, <)$ *there are an ordinal* α *and a bijection* $f : W \to \alpha$ *such that* $a < b$ *if and only if* $f(a) < f(b)$.

Before we move on to talking about cardinals we record some terminology about ordinals.

Definition 1.8. Let α be an ordinal.

- α is a **successor ordinal** if $\alpha = \beta + 1$ for some ordinal β.
- α is a **limit ordinal** if there is an infinite increasing sequence of ordinals $\langle \alpha_i \mid i < \lambda \rangle$ such that $\alpha = \bigcup_{i < \lambda} \alpha_i$.

Cardinals are special ordinals. The Axiom of Choice makes two possible definitions of cardinal equivalent.

Definition 1.9. An ordinal α is a **cardinal** if there is no surjection from an ordinal less than α onto α.

Clearly each $n \in \omega$ and ω itself are cardinals. We define $|A|$ to be the least ordinal α such that there is a bijection from A to α. $|A|$ is called the cardinality of A. It is not hard to see that $|A|$ is a cardinal. Note that $|A| = |B|$ if and only if there is a bijection from A to B. The following theorem makes it easier to prove that two sets have the same cardinality.

Theorem 1.10 (Cantor-Schroder-Bernstein). *If there are an injection from A to B and an injection from B to A, then there is a bijection from A to B.*

Proof. Without loss of generality we can take A and B to be disjoint, since we can replace A by $\{0\} \times A$ and B by $\{1\} \times B$. We let $f : A \to B$ and $g : B \to A$ be injections. We construct a bijection $h : A \to B$. Let $a \in A$ and define the set

$$S_a = \{\dots f^{-1}(g^{-1}(a)), g^{-1}(a), a, f(a), g(f(a)), \dots\}.$$

Let $b \in B$ and define the set

$$S_b = \{\dots g^{-1}(f^{-1}(b)), f^{-1}(b), b, g(b), f(g(b)) \dots\}.$$

Some note is due on these definitions. At some point we may be unable to take the inverse image. Suppose that $c \in A \cup B$ and S_c stops moving left because we cannot take an inverse image, if the left-most element of S_c is in A, then we call it A-terminating, otherwise we call it B-terminating.

Observe that if $c_1, c_2 \in A \cup B$ and $c_1 \in S_{c_2}$, then $S_{c_1} = S_{c_2}$.

Define h as follows. Let $a \in A$. If S_a is A-terminating or does not terminate, then define $h(a) = f(a)$. If S_a is B-terminating, then a is in the image of g, so define $h(a) = g^{-1}(a)$.

Clearly this defines a map from A to B, we just need to check that it is a bijection. First we check that it is onto. Let $b \in B$. If S_b is A-terminating or doesn't terminate, then b is in the image of f and $S_{f^{-1}(b)} = S_b$ is A-terminating or doesn't terminate, so we defined $h(f^{-1}(b)) = f(f^{-1}(b)) = b$ as required. If S_b is B-terminating, then $S_{g(b)}$ is also B-terminating, so we defined $h(g(b)) = g^{-1}(g(b)) = b$. It follows that h is onto.

Let $a_1, a_2 \in A$ and suppose that $h(a_1) = h(a_2)$. We will show that $a_1 = a_2$. If S_{a_1} and S_{a_2} are either

1. both A-terminating or non-terminating; or
2. both B-terminating,

then $a_1 = a_2$ follows from the injectivity of f or g.

Suppose for a contradiction that S_{a_1} is A-terminating or nonterminating and S_{a_2} is B-terminating. Then by the definition of h, $f(a_1) = h(a_1) = h(a_2) = g^{-1}(a_2)$. It follows that $S_{a_1} = S_{a_2}$ which is a contradiction. $\dashv$

We are now ready to introduce cardinal arithmetic.

Definition 1.11. Let κ and λ be cardinals.

- $\kappa + \lambda$ is the cardinality of $\{0\} \times \kappa \cup \{1\} \times \lambda$.
- $\kappa \cdot \lambda$ is the cardinality of $\kappa \times \lambda$.
- κ^λ is the cardinality of the set ${}^\lambda\kappa = \{f \mid f : \lambda \to \kappa\}$.

If κ and λ are infinite, then $\kappa + \lambda = \kappa \cdot \lambda = \max \kappa, \lambda$. Exponentiation turns out to be much more interesting. For any cardinal κ, $|\mathcal{P}(\kappa)| = 2^\kappa$.

THEOREM 1.12 (Cantor). *For any cardinal κ, $2^\kappa > \kappa$.*

PROOF. Suppose that there is a surjection H from κ onto 2^κ. Consider the function $f : \kappa \to 2$ given by $f(\alpha) = 0$ if and only if $H(\alpha)(\alpha) = 1$. (Recall that $H(\alpha)$ is a function from κ to 2.)

We claim that f is not in the range of H, a contradiction. Let $\alpha < \kappa$, then f is different from $H(\alpha)$, since $f(\alpha) = 0$ if and only if $H(\alpha)(\alpha) = 1$. $\dashv$

By Cantor's theorem we see that for any cardinal κ there is a strictly larger cardinal. We write κ^+ for the least cardinal greater than κ. Moreover the union of a collection of cardinals is a cardinal. The above facts allow us to use the ordinals to enumerate all of the cardinals.

1. $\aleph_0 = \omega$,
2. $\aleph_{\alpha+1} = \aleph_\alpha^+$ and
3. $\aleph_\gamma = \bigcup_{\alpha<\gamma} \aleph_\alpha$ for γ a limit ordinal.

We often write ω_α in place of $\aleph_\alpha$. They are the same object, but we think of ω_α in the context of ordinals and $\aleph_\alpha$ in the context of cardinals.

The Continuum Hypothesis (CH) states that $2^{\aleph_0} = \aleph_1$. From Cantor's theorem we know that $2^{\aleph_0} > \aleph_0$. CH is the assertion that the continuum $2^{\aleph_0}$ is the least cardinal greater than $\aleph_0$. The goal of the course is to prove that the axioms of ZFC cannot prove or disprove CH. To do this we will construct a model of ZFC where $2^{\aleph_0} = \aleph_1$ and a different model of ZFC where $2^{\aleph_0} = \aleph_2$. Don't worry if you don't know what this means, it will all be explained by the end of the course.

We now investigate a ZFC restriction on cardinal exponentiation.

DEFINITION 1.13. Let α be a limit ordinal. The **cofinality** of α, $\mathrm{cf}(\alpha)$ is the least $\lambda \leq \alpha$ such that there is an increasing sequence $\langle \alpha_i \mid i < \lambda \rangle$ of ordinals less than α such that $\sup_{i<\lambda} \alpha_i = \alpha$.

A sequence $\langle \alpha_i \mid i < \lambda \rangle$ of ordinals in α is said to be **cofinal in** α if $\sup_{i<\lambda} \alpha_i = \alpha$. Thus $\mathrm{cf}(\alpha)$ is the shortest length of an increasing sequence cofinal in α.

DEFINITION 1.14. A cardinal κ is **regular** if $\mathrm{cf}(\kappa) = \kappa$ and is **singular** otherwise.

PROPOSITION 1.15. $\mathrm{cf}(\alpha)$ *is a regular cardinal.*

THEOREM 1.16. *For any cardinal κ, $\kappa < \kappa^{\mathrm{cf}(\kappa)}$.*

PROOF. Set $\lambda = \mathrm{cf}(\kappa)$ and suppose that there is a surjection H from κ onto κ^λ. Fix an increasing sequence $\langle \alpha_i \mid i < \lambda \rangle$ which is increasing and cofinal in κ.

We define a function f which is not in the range of H, a contradiction. Let $f(i)$ be the least member of $\kappa \setminus \{H(\alpha)(i) \mid \alpha \leq \alpha_i\}$. Let $\alpha < \kappa$ and choose $i < \lambda$ such that $\alpha_i > \alpha$. It follows that $f(i) \neq H(\alpha)(i)$, so $f \neq H(\alpha)$. $\dashv$

Since $(2^\omega)^\omega = 2^{\omega \cdot \omega} = 2^\omega$, it follows that $\mathrm{cf}(2^\omega) > \omega$. In particular $2^\omega \neq \omega_\omega$.

§2. Cardinal characteristics. In this section we introduce some combinatorial notions. We start with a few examples of *cardinal characteristics of the continuum*.

Let $f, g \in {}^{\omega}\omega$. Recall that ${}^{\omega}\omega$ is the collection of functions from ω to ω. We define the notion of *eventual domination*, which is a weakening of the pointwise ordering. Let $f <^* g$ if and only if there is an $N < \omega$ such that for all $n \geq N$, $f(n) < g(n)$.

Clearly we can find an upperbound in this ordering for any finite collection of functions $\{f_0, \dots f_k\}$. For $n < \omega$ we define

$$f(n) = \max\{f_0(n), \dots f_k(n)\} + 1.$$

So in fact f is larger than each f_i on every coordinate. What happens if we allow our collection of functions to be countable, say $\{f_i \mid i < \omega\}$. Is it still possible to find a function f such that for all i, $f_i <^* f$?

The answer is Yes! To do this we use a diagonal argument. We know that on each coordinate we can only beat finitely many of the f_i. So we make sure that after the first n coordinates, we always beat the n^{th} function.

For $n < \omega$, we define

$$f(n) = \max_{i \leq n}(f_i(n)) + 1.$$

It is straightforward to check that this works. Now we ask if it is possible to continue, that is, to increase the size of our collection of functions to ω_1. Given $\{f_\alpha \mid \alpha < \omega_1\}$, can we find a single function f which eventually dominates each f_α?

The answer to this question is sensitive to the Set Theory beyond the Axioms of ZFC. For instance, if CH holds, then the answer is no, since all of ${}^{\omega}\omega$ can be enumerated in ω_1 steps. However, we will see that it is possible that the answer is yes if we assume Martin's Axiom.

We give a definition that captures the essence of this question.

Definition 2.1. Let $\mathfrak{b}$ be the least cardinal such that there exists a family of functions $\mathcal{F}$ with $|\mathcal{F}| = \mathfrak{b}$ such that no $f : \omega \to \omega$ eventually dominates all members of $\mathcal{F}$. Such a family is called an **unbounded family**.

$\mathfrak{b}$ is a cardinal characteristic of the continuum. We can phrase our observations as a theorem about $\mathfrak{b}$.

Theorem 2.2. *$\omega < \mathfrak{b} \leq 2^{\aleph_0}$.*

Our question about families of size ω_1 can now be rephrased as 'Is $\mathfrak{b} > \omega_1$?'.

We now introduce another cardinal characteristic $\mathfrak{a}$.

Definition 2.3. Let A, B be subsets of ω. We say that A and B are **almost disjoint** if $A \cap B$ is finite. A family $\mathcal{F}$ of infinite pairwise almost disjoint subsets of ω is **maximally almost disjoint** (MAD) if for any infinite subset B of ω, there is an $A \in \mathcal{F}$ such that $A \cap B$ is infinite.

An easy example of a MAD family is to take $\mathcal{F} = \{A, B\}$ where A is the set of odd natural numbers and B is the set of even natural numbers. In fact any partition of ω into finitely many pieces is a MAD family.

The following proposition is left as an exercise.

Proposition 2.4. *There is a MAD family of size* $2^{\aleph_0}$.

The following lemma is a part of what makes MAD families interesting.

Lemma 2.5. *There are no countably infinite MAD families.*

Proof. Let $\mathcal{F} = \{A_n \mid n < \omega\}$ be a countable family of pairwise almost disjoint sets. We will construct $B = \{b_n \mid n < \omega\}$ a subset of ω enumerated in increasing order. We ensure that b_{n+1} does not belong to any of $A_0, \dots A_n$. This ensures that $B \cap A_n$ is bounded by b_{n+1} (hence it is finite). To do this let b_0 be any member of A_0 and assuming that we have defined b_n for some n, let b_{n+1} be the least member of $A_{n+1} \setminus (A_0 \cup \cdots \cup A_n)$ greater than b_n. This is possible since the set in question is infinite by the almost disjointness of $\mathcal{F}$. ⊣

The definition of $\mathfrak{a}$ captures our questions about the possible sizes of MAD families.

Definition 2.6. Let $\mathfrak{a}$ be the least infinite cardinal such that there is a MAD family of size $\mathfrak{a}$.

So we have proved:

Theorem 2.7. $\omega < \mathfrak{a} \leq 2^{\aleph_0}$

It turns out that $\mathfrak{a}$ and $\mathfrak{b}$ are related.

Theorem 2.8 (Solomon, 1977). $\mathfrak{b} \leq \mathfrak{a}$.

Proof. It is enough to show that any almost disjoint family of size less than $\mathfrak{b}$ is not maximal. Let $\mathcal{F} = \{A_\alpha \mid \alpha < \kappa\}$ where $\kappa < \mathfrak{b}$ be an almost disjoint family. We may assume that the collection $\{A_n \mid n < \omega\}$ are pairwise disjoint.

We seek to define a useful collection of functions from ω to ω. Let $\omega \leq \alpha < \kappa$ and for $n < \omega$ define $f_\alpha(n)$ to be the least m such that the m^{th} member of A_n is larger than all elements in $A_n \cap A_\alpha$. This defines $\{f_\alpha \mid \omega \leq \alpha < \kappa\}$ and since $\kappa < \mathfrak{b}$ there is a function f which eventually dominates each f_α.

Now we define b_n to be the $f(n)^{th}$ member of A_n. Clearly $B = \{b_n \mid n < \omega\}$ is infinite and almost disjoint from each A_n, since it contains exactly one member from each A_n.

It remains to show that B is almost disjoint from each A_α for $\omega \leq \alpha < \kappa$. Fix α and let N be such that for all $n \geq N$, $f(n) > f_\alpha(n)$. For each $n \geq N$ we have that the $f(n)^{th}$ member of A_n is greater than all members of $A_n \cap A_\alpha$, since $f(n) > f_\alpha(n)$. In particular b_n, which is the $f(n)^{th}$ member of A_n is not in A_α for all $n \geq N$. So B works. ⊣

§3. Martin's Axiom. We need some definitions in order to formulate Martin's Axiom.

DEFINITION 3.1. A **partially ordered set** (**poset**) is a pair $(\mathbb{P}, \leq)$ where $\leq$ is a binary relation on $\mathbb{P}$ such that $\leq$ is

1. reflexive; for all $p \in \mathbb{P}$, $p \leq p$,
2. transitive; for all $p, q, r \in \mathbb{P}$, if $p \leq q$ and $q \leq r$, then $p \leq r$, and
3. antisymmetric; for all $p, q \in \mathbb{P}$, if $p \leq q$ and $q \leq p$, then $p = q$.

We also require that our posets have a unique maximal element $\mathbb{1}_{\mathbb{P}}$, i.e. for all $p \in \mathbb{P}$, $p \leq \mathbb{1}_{\mathbb{P}}$.

For simplicity, we will always refer to 'the poset $\mathbb{P}$' instead of the poset $(\mathbb{P}, \leq)$. Elements of $\mathbb{P}$ are often called **conditions** and when $p \leq q$ we say that p is an extension (or strengthening) of q. Posets are everywhere and we will see many examples throughout the course.

As a running example we will consider the set $\mathbb{P} = \{p \mid p : n \to 2\}$ ordered by $p_1 \leq p_2$ if and only if $p_1 \supseteq p_2$. It is not hard to check that this is a poset.

DEFINITION 3.2. Let $\mathbb{P}$ be a poset and $p, q \in \mathbb{P}$.

1. p and q are **comparable** if $p \leq q$ or $q \leq p$.
2. p and q are **compatible** if there is an $r \in \mathbb{P}$ such that $r \leq p, q$.

Incomparable and *incompatible* mean 'not comparable' and 'not compatible' respectively.

DEFINITION 3.3. Let $\mathbb{P}$ be a poset and $A \subseteq \mathbb{P}$. A is an **antichain** if any two elements of A are incompatible.

Note that for a fixed $n < \omega$ the collection $\{p \mid \mathrm{dom}(p) = n\}$ is an antichain in our example poset.

DEFINITION 3.4. Let $\mathbb{P}$ be a poset. $\mathbb{P}$ has the **countable chain condition** (is ccc) if every antichain of $\mathbb{P}$ is countable.

Our example poset is ccc for trivial reasons; the whole poset is countable.

DEFINITION 3.5. Let $\mathbb{P}$ be a poset. A subset $D \subseteq \mathbb{P}$ is **dense** if for all $p \in \mathbb{P}$ there is $q \in D$ such that $q \leq p$.

In our running example both of the following sets are dense for any $n < \omega$: $\{p \in \mathbb{P} \mid \mathrm{dom}(p) > n\}$ and $\{p \in \mathbb{P} \mid \mathrm{dom}(p) \text{ is even}\}$. How are these different?

DEFINITION 3.6. Let $\mathbb{P}$ be a poset. A subset $D \subseteq \mathbb{P}$ is **open** if for all $p \in D$ and for all $q \leq p$, $q \in D$.

The first of the two sets above is open and the second is not.

DEFINITION 3.7. A subset $G \subseteq \mathbb{P}$ is a **filter** if

1. for all $p \in G$ and $q \geq p$, $q \in G$, and
2. for all $p, q \in G$ there is $r \in G$ with $r \leq p, q$.

If $\mathcal{D}$ is a collection of dense subsets of $\mathbb{P}$, then we say that G is $\mathcal{D}$-generic if for every $D \in \mathcal{D}$, $D \cap G \neq \varnothing$.

We are now ready to formulate Martin's Axiom.

Definition 3.8. MA(κ) is the assertion that for every ccc poset $\mathbb{P}$ and collection of κ-many dense sets $\mathcal{D}$, there is a $\mathcal{D}$-generic filter over $\mathbb{P}$.

MA is the assertion that MA(κ) holds for all $\kappa < 2^{\aleph_0}$. Roughly speaking MA asserts that if an object has a reasonable collection of approximations, then it exists.

Proposition 3.9. MA(ω) *holds even if we drop the ccc requirement.*

Proof. Let $\mathcal{D} = \{D_n \mid n < \omega\}$ be a collection of dense subsets of a poset $\mathbb{P}$. We construct a decreasing sequence $\langle p_n \mid n < \omega\rangle$ such that $p_n \in D_n$ for all n. Let $p_0 \in D_0$. Suppose we have constructed p_n for some $n < \omega$. We choose $p_{n+1} \in D_{n+1}$ with $p_{n+1} \leq p_n$ by density.

We define $G = \{p \in \mathbb{P} \mid p \geq p_n \text{ for some } n < \omega\}$. It is not hard to see that G is a $\mathcal{D}$-generic filter over $\mathbb{P}$. $\dashv$

Proposition 3.10. *If* MA(κ) *holds, then* $\kappa < 2^{\aleph_0}$. *In particular* MA($2^{\aleph_0}$) *fails.*

Proof. Suppose that MA(κ) holds. It is enough to show that given a collection $\{f_\alpha \mid \alpha < \kappa\}$ of functions from ω to 2, there is a function g which is not equal to any f_α.

Let $\mathbb{P}$ be as in our running example. We claim that for each $\alpha < \kappa$, the set $E_\alpha = \{p \mid \text{for some } n \in \text{dom}(p)\ f_\alpha(n) \neq p(n)\}$ is dense. Given a $p \in \mathbb{P}$ choose an $n \in \omega \setminus \text{dom}(p)$ and consider the condition $p \cup \{\langle n, f_\alpha(n) +_2 1\rangle\}$, which is in E_α.

We also need $D_n = \{p \mid \text{dom}(p) > n\}$ which is dense as we discussed. We let $\mathcal{D} = \{D_n \mid n < \omega\} \cup \{E_\alpha \mid \alpha < \kappa\}$ and apply MA(κ) to obtain G.

We claim that $g = \bigcup G$ is a function. For if $\langle n, y\rangle$ and $\langle n, y'\rangle$ are both in g, then there are $p, p' \in G$ so that $p(n) = y$ and $p'(n) = y'$. Since G is a filter, there is some $q \leq p, p'$; so $y = q(n) = y'$, as needed. Since $G \cap D_n \neq \varnothing$ for each $n < \omega$, we have that g has domain ω; and since $G \cap E_\alpha \neq \varnothing$ for each $\alpha < \kappa$, we have $g \neq f_\alpha$. $\dashv$

Proposition 3.11. MA($\aleph_1$) *fails if we remove the ccc requirement.*

Proof. Let $\mathbb{P} = \{p \mid p : n \to \omega_1 \text{ for some } n < \omega\}$ ordered by $p_1 \leq p_2$ if and only if $p_1 \supseteq p_2$. We define $E_\alpha = \{p \mid \alpha \in \text{ran}(p)\}$ and $D_n = \{p \mid n \in \text{dom}(p)\}$. It is not hard to see that these sets are dense.

Let G be generic for all of our dense sets. We have arranged that $g = \bigcup G$ is a surjection from ω onto ω_1. Such a function cannot exist. $\dashv$

The way we have formulated MA, CH implies that MA holds for trivial reasons. It is consistent with ZFC that MA holds with the continuum large, but this result is beyond the scope of the course. The reason that we introduce MA is that its statement and applications allow us to get acquainted with core machinery of forcing (posets, filters, etc.) without being burdened by the metamathematical complications of forcing proper (which we will tackle separately soon enough).

§4. Applications of MA **to cardinal characteristics.** We continue our applications of MA by showing how MA influences cardinal characteristics of the continuum. We can view these applications as extensions of the diagonalization arguments we used to show that $\mathfrak{b}$ and $\mathfrak{a}$ are uncountable.

We will prove the following theorem.

Theorem 4.1. MA *implies* $\mathfrak{b} = 2^{\aleph_0}$.

Using Solomon's Theorem we have,

Corollary 4.2. MA *implies* $\mathfrak{a} = 2^{\aleph_0}$.

Given a collection of functions of size less than continuum we need to build a ccc poset which approximates a function f which dominates all of the functions in our collection. In order to satisfy the ccc requirement our approximations will be finite.

Proof. We define a poset $\mathbb{P}$ to be the collection of pairs (p, A) where $p \in {}^{<\omega}\omega$ and A is a finite subset of ${}^{\omega}\omega$. For the ordering we set $(p, A) \leq (q, B)$ if and only if $p \supseteq q$, $A \supseteq B$ and for all $f \in B$ and all $n \in \mathrm{dom}(p) \setminus \mathrm{dom}(q)$, $p(n) > f(n)$. (You should check that $\leq$ is transitive.) The p-part of the condition is growing the function from ω to ω and the A-part is a collection of functions which we promise to dominate when we extend the p-part. The poset $\mathbb{P}$ is called the **dominating poset**.

We claim that $\mathbb{P}$ is ccc. It is enough to show that every set of conditions of size ω_1 contains two pairwise compatible conditions. Let $\langle (p_\alpha, A_\alpha) \mid \alpha < \omega_1 \rangle$ be a sequence of conditions in $\mathbb{P}$. By the pigeonhole principle there is an unbounded set $I \subseteq \omega_1$ such that for all $\alpha, \beta \in I$, $p_\alpha = p_\beta$.

Let $\alpha, \beta \in I$ and define $p = p_\alpha = p_\beta$. We claim that $(p, A_\alpha \cup A_\beta)$ is a lower bound for both (p_α, A_α) and (p_β, A_β). This is clear, since the third condition for extension is vacuous. So we have actually shown that that given a sequence of ω_1-many conditions in $\mathbb{P}$ there is a subsequence of ω_1-many conditions which are pairwise compatible. This property is called the **ω_1-Knaster property**.

We will apply MA to this poset. Let $\mathcal{F} = \{f_\alpha \mid \alpha < \kappa\}$ be a collection of functions from ω to ω where κ is some cardinal less than $2^{\aleph_0}$. Now we need a collection of dense sets to which we will apply MA. First, we have for each $n < \omega$, the collection $D_n = \{(p, A) \mid n \in \mathrm{dom}(p)\}$. Given a condition (p, A) we can just extend the p to have n in the domain ensuring that we choose a value larger than the maximum of the finitely many functions in A on each coordinate we add. Call the extension q. It is clear that $(q, A) \leq (p, A)$ and $(q, A) \in D_n$. So D_n is dense.

For each $\alpha < \kappa$ we define $E_\alpha = \{(p, A) \mid f_\alpha \in A\}$. Clearly this is dense, since given a condition (q, B), $(q, B \cup \{f_\alpha\}) \leq (q, B)$ and is a member of E_α.

From here the proof is easy. By MA we can choose G a $\mathcal{D}$-generic filter where $\mathcal{D} = \{D_n \mid n < \omega\} \cup \{E_\alpha \mid \alpha < \kappa\}$. Let $f = \bigcup \{p \mid (p, A) \in G \text{ for some } A\}$. By the usual argument, $f \in {}^{\omega}\omega$. To see that f eventually dominates each f_α, let $\alpha < \kappa$ and choose a condition $(p, A) \in G \cap E_\alpha$. Let $N = \mathrm{dom}(p)$. We claim that for all $n \geq N$, $f(n) > f_\alpha(n)$. Fix such an n and choose a condition $(q, B) \in G \cap D_n$ with $(q, B) \leq (p, A)$. By the definition of extension $q(n) > f_\alpha(n)$, but $q(n) = f(n)$ so we are done. ⊣

We sketch another very similar application of MA and leave some of the details as exercises.

Theorem 4.3. *Assume* MA(κ) *and let* $\mathcal{A}$ *and* $\mathcal{C}$ *be collections of size* $\leq \kappa$ *of subsets of* ω *such that for every* $y \in \mathcal{C}$ *and every finite* $F \subseteq \mathcal{A}$ *the set* $y \setminus \bigcup F$ *is infinite. There is a single subset* $Z \subseteq \omega$ *such that* $X \cap Z$ *is finite for all* $X \in \mathcal{A}$ *and* $Y \cap Z$ *is infinite for* $Y \in \mathcal{C}$*.*

The proof is very similar to the previous so we will define the poset and leave the rest as an exercise. Let $\mathbb{P}$ be the collection of pairs (s, F) where $s \in [\omega]^{<\omega}$ and $F \subseteq \mathcal{A}$ is finite. Let $(s_0, F_0) \leq (s_1, F_1)$ if and only if $s_0 \supseteq s_1$, $F_0 \supseteq F_1$ and for all $n \in s_0 \setminus s_1$, $n \notin \bigcup F_1$.

Most of the proof is as before. Here is a helpful hint: Show that for each $n < \omega$ and $Y \in \mathcal{C}$, the set $E^n_Y = \{(s, F) \mid \text{there is } m \geq n \text{ such that } m \in s \cap Y\}$ is dense.

Corollary 4.4. MA *implies* $\mathfrak{a} = 2^{\aleph_0}$

Apply the previous theorem with $\mathcal{C} = \{\omega\}$.

Corollary 4.5. *Suppose that* MA(κ) *holds. If* $\mathcal{B}$ *is an almost disjoint family of size* κ *and* $\mathcal{A} \subseteq \mathcal{B}$*, then there is a* Z *which has infinite intersection with each member of* $\mathcal{B} \setminus \mathcal{A}$ *and finite intersection with each member of* $\mathcal{A}$*.*

Just apply the theorem with $\mathcal{A}$ as itself and $\mathcal{C} = \mathcal{B} \setminus \mathcal{A}$. Note that the set Z codes the set $\mathcal{A}$ in that if we are given Z we can define $\mathcal{A} = \{A \in \mathcal{B} \mid Z \cap A \text{ is finite}\}$. This gives us the following fact.

Theorem 4.6. MA *implies for all infinite* $\kappa < 2^{\aleph_0}$, $2^\kappa = 2^{\aleph_0}$.

Proof. Let $\mathcal{B}$ be an almost disjoint family of size κ. It is enough to show that $|\mathcal{P}(\mathcal{B})| = 2^{\aleph_0}$.

Define $\Gamma : \mathcal{P}(\omega) \to \mathcal{P}(\mathcal{B})$ by $\Gamma(Z) = \{A \in \mathcal{B} \mid A \cap Z \text{ is finite}\}$. Γ is surjective by the previous corollary. $\dashv$

Corollary 4.7. MA *implies* $2^{\aleph_0}$ *is regular.*

Proof. Suppose $\mathrm{cf}(2^{\aleph_0}) = \kappa < 2^{\aleph_0}$. Then we have

$$(2^{\aleph_0})^\kappa = (2^\kappa)^\kappa = 2^\kappa = 2^{\aleph_0}$$

which violates König's Lemma, a contradiction. $\dashv$

§5. Applications of MA to Lebesgue measure. Another application of MA is to Lebesgue measure. To begin we recall some facts about Lebesgue measure. Lebesgue measure assigns a size to certain sets of real numbers. We begin by trying to extend the notion of the length of an interval. We first define a notion of **outer measure** on all sets of real numbers. Given $A \subseteq \mathbb{R}$ we define

$$\mu^*(A) = \inf\left\{ \sum_{n<\omega}(b_n - a_n) \;\middle|\; A \subseteq \bigcup_{n<\omega}(a_n, b_n)\right\}.$$

We list some properties of this outer measure. These properties will be true of the full Lebesgue measure as well.

Proposition 5.1. *μ^* has the following properties:*

1. $\mu^*(\varnothing) = 0$.
2. *For all* $E \subseteq F$, $\mu^*(E) \subseteq \mu^*(F)$.
3. *For all* $\{E_n | n < \omega\}$, $\mu^*(\bigcup_{n<\omega} E_n) \leq \sum_{n<\omega} \mu^*(E_n)$.

Proof. The first item is clear. For the second, notice that any open cover of F is also an open cover of E. The main point is the third item. Let $\epsilon > 0$. By the definition of μ^* for each n we can choose an open set U_n such that $\mu^*(U_n) \leq \mu^*(E_n) + \epsilon \cdot 2^{-n-1}$.

Note that $\bigcup_{n<\omega} U_n$ is an open set covering $E = \bigcup_{n<\omega} E_n$. So we have

$$\mu^*(E) \leq \sum_{n<\omega} \mu^*(U_n) \leq \sum_{n<\omega} \left(\mu^*(E_n) + \epsilon \cdot 2^{-n-1}\right) = \sum_{n<\omega} \mu^*(E_n) + \epsilon.$$

Since ϵ was arbitrary we have the result. $\dashv$

It is not hard to see that μ^* returns the length of an interval, that is $\mu^*(a, b) = b - a$. Further, recall that an open subset of the real line U can be written uniquely as the union of countably many disjoint open intervals. (To do this let I_x be the union of all open intervals contained in U with x as a member. If $I_x \neq I_y$, then $I_x \cap I_y = \varnothing$. So $\bigcup_{x\in U} I_x = U$ is a disjoint union of open intervals and hence there can only be countably many intervals involved.) So if we write $U = \bigcup_{n<\omega}(a_n, b_n)$ where the intervals are pairwise disjoint, then it is clear that we have $\mu^*(U) = \sum_{n<\omega}(b_n - a_n)$.

It turns out that the outer measure μ^* is poorly behaved on arbitrary sets. To define the full Lebesgue measure we want to restrict ourselves to certain nice sets. For this, we introduce the **Borel sets**. The collection of Borel sets $\mathcal{B}$ is the smallest set which contains the open sets and is closed under countable unions and complements. (A set closed under countable unions and complements is called a σ-algebra.)

As an aside, note the Borel sets can obtained by defining

$$\mathcal{B} = \bigcap\{\mathcal{A} \mid \mathcal{A} \text{ is a } \sigma\text{-algebra containing the open sets}\}.$$

But we can also give a more concrete description of these sets as follows. Define $\mathbf{\Sigma}^0_1$ the be the collection of open sets. Then put, for $\alpha < \omega_1$,

$$\mathbf{\Pi}^0_\alpha = \{X \mid X \text{ is a complement of some } Y \in \mathbf{\Sigma}^0_\alpha\},$$
$$\mathbf{\Sigma}^0_\alpha = \{X \mid X \text{ is a countable union of sets in } \textstyle\bigcup_{\xi<\alpha} \mathbf{\Pi}^0_\alpha\}.$$

It can then be shown that $\mathcal{B} = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha$.

We are now ready to define what it means to be Lebesgue measurable.

Definition 5.2. A set $A \subseteq \mathbb{R}$ is **Lebesgue measurable** if there is a Borel set B such that $\mu^*(A \triangle B) = 0$. In this case the **Lebesgue measure of** A is $\mu(A) = \mu^*(A)$. We call the collection of Lebesgue measurable sets $\mathcal{L}$.

We catalog some properties of Lebesgue measure.

Proposition 5.3. *$\mathcal{L}$ is the smallest σ-algebra containing the Borel sets and the sets of outer measure zero.*

Proposition 5.4. *$\mathcal{B} \neq \mathcal{L}$.*

Theorem 5.5. *$\mathcal{L}$ and μ have the following properties:*

1. *(Monotonicity) If $A, B \in \mathcal{L}$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$.*
2. *(Translation invariance) If $A \in \mathcal{L}$ and $t \in \mathbb{R}$, then $t + A = \{t + x \mid x \in A\} \in \mathcal{L}$ and $\mu(A) = \mu(t + A)$.*
3. *(Countable additivity) If $\{A_n \mid n < \omega\} \subseteq \mathcal{L}$ is a collection of pairwise disjoint sets, then $\mu(\bigcup_{n<\omega} A_n) = \sum_{n<\omega} \mu(A_n)$.*

Theorem 5.6 (AC). *There is $A \subseteq \mathbb{R}$ with $A \notin \mathcal{L}$.*

Proof. Define an equivalence relation on $\mathbb{R}$ by $x \sim y$ if and only if $|x - y|$ is rational. Note each equivalence class is countable. Let F be a choice function for the equivalence classes; we can further assume $F([x]_\sim) \in [0, 1]$ for each x.

Let A be the range of F. We claim A is not Lebesgue measurable. Notice $\mathbb{R} = \bigcup_{q\in\mathbb{Q}}(A + q)$, so by countable additivity, we have $0 < \mu(A)$. But then $\mu(\bigcup_{n\in\omega} A + \frac{1}{n+1})$ is a subset of $[0, 2]$ with infinite measure, a contradiction. $\dashv$

Our application of MA will be to sets of measure zero and will generalize the following fact which is an easy consequence of countable sub-additivity.

Proposition 5.7. *The union of countably many measure zero sets has measure zero.*

For ease of notation we let $\mathcal{C}$ be the collection of finite unions of open intervals with rational endpoints. Note that $\mathcal{C}$ is countable. We will show that open sets can be approximated closely in measure by members of $\mathcal{C}$.

Proposition 5.8. *Let U be an open set with $0 < \mu(U) < \infty$. For every $\epsilon > 0$ there is a member $Y \in \mathcal{C}$ such that $Y \subseteq U$ and $\mu(U \setminus Y) < \epsilon$.*

Proof. Let $\epsilon > 0$ and assume that $\mu(U)$ is some positive real number m. Write $U = \bigcup_{n<\omega}(a_n, b_n)$ where the collection $\{(a_n, b_n) \mid n < \omega\}$ is pairwise disjoint. We choose $N < \omega$ such that $\sum_{n\geq N}(b_n - a_n) < \frac{\epsilon}{2}$. For each $n < N$ we choose rational numbers q_n, r_n such that $a_n < q_n < r_n < b_n$ and

$$\mu((a_n, b_n) \setminus (q_n, r_n)) = |b_n - r_n| + |q_n - a_n| < \frac{\epsilon}{2} \cdot 2^{-n-1}$$

We set $Y = \bigcup_{n<N}(q_n, r_n) \in \mathcal{C}$. An easy calculation shows that this works. $\dashv$

We are ready for our application of MA to Lebesgue measure.

Theorem 5.9. MA(κ) *implies the union of κ-many measure zero sets is measure zero.*

Proof. Let $\epsilon > 0$. Define a poset $\mathbb{P}$ to be the collection of open $p \in \mathcal{L}$ such that $\mu(p) < \epsilon$ and set $p_0 \leq p_1$ if and only if $p_0 \supseteq p_1$. As usual we need to show that $\mathbb{P}$ is ccc.

Towards showing that $\mathbb{P}$ is ccc, we let $\{p_\alpha \mid \alpha < \omega_1\}$ be a collection of conditions from $\mathbb{P}$. For each α we know that $\mu(p_\alpha) < \epsilon$, so there is an $n_\alpha < \omega$ such that $\mu(p_\alpha) < \epsilon - \frac{1}{n_\alpha}$. By the pigeonhole principal we may assume that there is an n such that $n = n_\alpha$ for all $\alpha < \omega_1$.

Now for each α we choose $Y_\alpha \in \mathcal{C}$ such that $Y_\alpha \subseteq p_\alpha$ and $\mu(p_\alpha \setminus Y_\alpha) < \frac{1}{2n}$. Since $\mathcal{C}$ is countable we may assume that there is a $Y \in \mathcal{C}$ such that $Y = Y_\alpha$ for all $\alpha < \omega_1$. Now let $\alpha < \beta < \omega_1$, we have

$$\mu(p_\alpha \cup p_\beta) \leq \mu(p_\alpha \setminus Y) + \mu(p_\beta \setminus Y) + \mu(Y) < \frac{1}{2n} + \frac{1}{2n} + \epsilon - \frac{1}{n} = \epsilon.$$

So p_α and p_β are compatible.

We use this poset to prove the theorem. Let $\{A_\alpha \mid \alpha < \kappa\}$ be a collection of measure zero sets. We want to show that the measure of the union is zero. Let $\epsilon > 0$ and $\mathbb{P}$ be defined as above. We claim that $E_\alpha = \{p \in \mathbb{P} \mid A_\alpha \subseteq p\}$ is dense for each $\alpha < \kappa$. Let $q \in \mathbb{P}$. Since $\mu(A_\alpha) = 0$ we can find an open set r such that $A_\alpha \subseteq r$ and $\mu(r) < \epsilon - \mu(q)$. Clearly $p = q \cup r \in E_\alpha$. So E_α is dense.

Now we apply MA to $\mathbb{P}$ and the collection of $\{E_\alpha \mid \alpha < \kappa\}$ to obtain G. We claim that $U = \bigcup G$ is an open set containing the union of the A_α and $\mu(U) \leq \epsilon$. Clearly U is open since it is the union of open sets. Clearly it contains the union of the A_α, since G meets each E_α. It remains to show that $\mu(U) \leq \epsilon$.

We claim that if $\{p_n \mid n < \omega\}$ is a subset of G, then $\mu(\bigcup_{n<\omega} p_n) \leq \epsilon$. Note that since each $p_n \in G$, $p_0 \cup \cdots \cup p_n \in G$. Hence $\mu(p_0 \cup \cdots \cup p_n) < \epsilon$. If we define $q_n = p_n \setminus (p_0 \cup \cdots \cup p_{n-1})$, then we have $\mu(q_0 \cup \cdots \cup q_n) = \mu(p_0 \cup \cdots \cup p_n) < \epsilon$. So we have

$$\mu\left(\bigcup_{n<\omega} p_n\right) = \mu\left(\bigcup_{n<\omega} q_n\right) = \sum_{n<\omega} \mu(q_n) \leq \epsilon$$

since each partial sum is less than ϵ. This finishes the claim.

To finish the proof it is enough to show that there is a countable subset $B \subseteq G$ such that $\bigcup B = U$. Suppose that $x \in U$. Then $x \in p$ for some $p \in G$. So we can find $q_x \in \mathcal{C}$ such that $x \in q_x \subseteq p$. Since G is a filter $q_x \in G$. So $G = \bigcup_{x \in U} q_x$. But $\mathcal{C}$ is countable so $B = \{q_x \mid x \in U\}$ is as required. $\dashv$

§6. Applications of MA to ultrafilters. Ultrafilters are an important concept in modern set theory. We introduce ultrafilters in some generality and then give an application of MA to ultrafilters on ω.

DEFINITION 6.1. Let X be a set. A collection $\mathcal{F} \subseteq \mathcal{P}(X)$ is a **filter on** X if all of the following properties hold:

1. $X \in \mathcal{F}$ and $\varnothing \notin \mathcal{F}$.
2. If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$
3. If $A \subseteq B$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$.

DEFINITION 6.2. A filter $\mathcal{F}$ on X is **principal** if there is a set $X_0 \subseteq X$ such that $\mathcal{F} = \{A \subseteq X \mid X_0 \subseteq A\}$. Otherwise $\mathcal{F}$ is **nonprincipal**.

As an example we can always define a filter on a cardinal κ by setting $\mathcal{F} = \{A \subseteq \kappa \mid \kappa \setminus A \text{ is bounded in } \kappa\}$. One way to think about a filter is to think of members of the filter as 'large'. If $\kappa = \omega$, then the filter that we just defined is called the **Fréchet filter**.

We want to know when a collection of sets can be extended to a filter. The following definition gives a sufficient condition.

DEFINITION 6.3. A collection of sets $\mathcal{A} \subseteq \mathcal{P}(X)$ has the **finite intersection property** if for all $A_0, \dots A_n$ from $\mathcal{A}$, $\bigcap_{i \leq n} A_i$ is nonempty.

PROPOSITION 6.4. *If $\mathcal{A} \subseteq \mathcal{P}(X)$ has the finite intersection property then there is a filter on X containing $\mathcal{A}$.*

PROOF. Suppose $\mathcal{A}$ has the finite intersection property; define $\mathcal{F} = \{B \subseteq X \mid B \supseteq A_0 \cap \dots \cap A_n \text{ for some } A_1, \dots A_n \in \mathcal{A}\}$. It's easy to check $\mathcal{F}$ is a filter. ⊣

The filter in the above proof is called the filter **generated by** $\mathcal{A}$.

DEFINITION 6.5. A filter $\mathcal{F}$ on X is an **ultrafilter** if for every $A \subseteq X$ either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$. A filter $\mathcal{F}$ on X is **maximal** if there is no filter $\mathcal{F}'$ on X which properly contains $\mathcal{F}$.

PROPOSITION 6.6. *A filter is maximal if and only if it is an ultrafilter.*

PROOF. If $\mathcal{F}$ is an ultrafilter, then any set $A \notin \mathcal{F}$ must have $X \setminus A \in \mathcal{F}$. It follows that any proper extension of $\mathcal{F}$ cannot have the finite intersection property, so cannot be a filter. This shows $\mathcal{F}$ is maximal.

Conversely, if $\mathcal{F}$ is maximal, let $A \subseteq X$; one of $\mathcal{F} \cup \{A\}$ and $\mathcal{F} \cup \{X \setminus A\}$ has the finite intersection property. (Why?) One of these extensions can be further extended to a filter. Since $\mathcal{F}$ is maximal, we must have either A or its complement in $\mathcal{F}$. ⊣

We have an easy example of an ultrafilter: for any $x \in X$, let $\mathcal{F}$ be the principal filter generated by x. Indeed, all principal ultrafilters are of this form. The following gives a more interesting class of ultrafilters.

PROPOSITION 6.7. *Every filter can be extended to an ultrafilter.*

PROOF. Given a filter $\mathcal{F}$, consider the partial order of filters containing $\mathcal{F}$ ordered by inclusion. It's easy to check that the union of any chain of filters is a filter, so by Zorn's lemma, there is a maximal such filter, and by the previous proposition, this is an ultrafilter. ⊣

We saw the Fréchet filter was $\mathcal{F} = \{A \subseteq \omega \mid \omega \setminus A \text{ is finite}\}$. Now $\mathcal{F}$ can be extended to an ultrafilter $\mathcal{U}$. $\mathcal{U}$ is nonprincipal since it contains the complement of every singleton.

Our application of MA to ultrafilters will be to construct a special kind of ultrafilter called a Ramsey ultrafilter. To motivate the definition we recall the following theorem.

THEOREM 6.8 (Ramsey). *For every $\chi : [\omega]^2 \to 2$, there is an infinite set B such that χ is constant on $[B]^2$.*

PROOF. We construct three sequences A_i, ϵ_i, a_i such that $A_{i+1} \subseteq A_i$, $a_i < a_{i+1}$ and $\epsilon_i \in 2$ for all $i < \omega$. Let $a_0 = 0$ and $A_0 = \omega$. For the induction step, suppose that we have defined A_n, a_n for some $n < \omega$. We choose $\epsilon_n \in 2$ such that $A_{n+1} = \{k \in A_n \setminus (a_n + 1) \mid \chi(a_n, k) = \epsilon_n\}$ is infinite and let a_{n+1} be the least member of A_{n+1}. This completes the construction.

Let $I \subseteq \omega$ be infinite and $\epsilon \in 2$ such that for all $i \in I$, $\epsilon_i = \epsilon$. We set $B = \{a_i \mid i \in I\}$ and claim that χ is constant on $[B]^2$. Suppose $a_i < a_j$ are in B. Then $a_j \in A_{i+1}$ and so $\chi(a_i, a_j) = \epsilon_i = \epsilon$ as required. $\dashv$

The set B we constructed is often called **monochromatic**. Here is a sample application of Ramsey's theorem.

THEOREM 6.9 (Bolzano-Weierstrass). *Every sequence of real numbers has a monotone subsequence.*

PROOF. Let $\langle a_n \mid n < \omega \rangle$ be a sequence of real numbers. We define a coloring $\chi : [\omega]^2 \to 2$ by $\chi(m, n) = 0$ if $a_m \leq a_n$ and $\chi(m, n) = 1$ otherwise. (Whenever we define a coloring we think of the domain as pairs (m, n) with $m < n$.)

By Ramsey's theorem there is an infinite $B \subseteq \omega$ such that B is monochromatic for χ. It's easy to see that $\langle a_n \mid n \in B \rangle$ is monotone. $\dashv$

DEFINITION 6.10. An ultrafilter $\mathcal{U}$ on ω is **Ramsey** if for every coloring $\chi : [\omega]^2 \to 2$, there is $B \in \mathcal{U}$ such that χ is constant on $[B]^2$.

Note that a Ramsey ultrafilter must be nonprincipal. For let $n < \omega$ and define χ as follows. If $k > n$, then we set $\chi(n, k) = 0$ and for all other pairs $l < k$, we set $\chi(l, k) = 1$. Clearly n cannot take part in any monochromatic set for χ.

THEOREM 6.11. MA *implies there is a Ramsey ultrafilter.*

PROOF. There are 2^ω possible colorings and we want to construct an ultrafilter with a monochromatic set for each coloring. We enumerate all of the colorings $\langle \chi_\alpha \mid \alpha < 2^\omega \rangle$ and construct a tower $T = \{A_\alpha \mid \alpha < 2^\omega\}$ such that A_α is monochromatic for χ_α. Recall that a tower of subsets of ω has the property that for all $\alpha < \beta$, $A_\beta \subseteq^* A_\alpha$.

Suppose that we have constructed A_α for each $\alpha < \beta$. Since MA implies $\mathfrak{t} = 2^\omega$, we can find $A \subseteq^* A_\alpha$ for all $\alpha < \beta$. By Ramsey's theorem we can find an infinite subset A_β of A which is monochromatic for χ_β. This completes the construction.

To complete the proof we notice that our Tower T has the finite intersection property! Hence T can be extended to an ultrafilter $\mathcal{U}$ which is clearly Ramsey. $\dashv$