§11. The independence of CH. As advertised we will prove the independence of CH from the axioms of ZFC. As we saw when we discussed formal proofs and model theory, it is enough to construct two models of ZFC, one in which CH holds and the other in which it fails. Of course, to do so we will need to assume the consistency of ZFC in addition to the axioms of ZFC, since otherwise there may not even be a model of ZFC to begin with; thus the independence of CH is a *relative consistency result*, in the sense that if ZFC is consistent, then so are each of the theories $\mathrm{ZFC} + \mathrm{CH}$ and $\mathrm{ZFC} + \neg\,\mathrm{CH}$.

In fact, we use a bit more than consistency: We will assume that there is a *transitive* model of ZFC, which a bit stronger than just the existence of a model of ZFC. This assumption can be done away with, however, by dealing with large enough finite fragments of ZFC and using the reflection theorem.

For example, suppose towards a contradiction $\mathrm{ZFC} \vdash \mathrm{CH}$; then there is some finite fragment T of ZFC so that $T \vdash \mathrm{CH}$. Using the forcing theorems, we know there is a finite theory T' so that whenever M is a countable transitive model of T', $\mathbb{P} \in M$, and G is $\mathbb{P}$-generic over M, then $M[G]$ satisfies T (essentially, T' needs to be large enough to ensure existence of the appropriate names for objects and that the needed instances of the forcing theorems hold in M). Furthermore (as we see later) an appropriate choice of poset $\mathbb{P}$ will ensure $M[G] \models \neg\,\mathrm{CH}$. But then by the reflection and Mostowski Collapse theorems, there is a countable transitive model M of T', hence also a model $M[G]$ of $T + \neg\,\mathrm{CH}$, contradicting our choice of T.

So in all of our forcing arguments we will just work with a transitive model of full ZFC, because we know that there is a standard way to do without it.

One more note on a common theme in forcing arguments. In general it is a bad idea to collapse ω_1. What is meant by this is we do not want to pass to a generic extension $M[G]$ in which there is a function $f : \omega \to \omega_1^M$ which is surjective. From the point of view of such an extension $M[G]$, the ω_1 in M is a countable ordinal, and this is precisely the sort of disturbance to the universe of M that we wish to avoid.

We will see two methods for arguing that ω_1 is not collapsed. The key idea is to prove some property of the poset used in forcing. The first idea which we have already seen is the notion of *chain condition*. The second idea which we have not yet seen is the notion of *closure*.

11.1. The consistency of CH. We wish to construct a model of $\mathrm{ZFC} + \mathrm{CH}$ by forcing. Given a countable transitive model M of ZFC we describe a poset $\mathbb{P}$ such that whenever G is $\mathbb{P}$-generic, $\omega_1^M = \omega_1^{M[G]}$ and $M[G] \vDash \mathrm{CH}$. The poset is easy to describe. We let $\mathbb{P} = \{p \mid p : \alpha \to 2 \text{ for some countable ordinal } \alpha\}$ ordered by extension, i.e. $p_1 \leq p_2$ if and only if $p_1 \supseteq p_2$.

To show that ω_1 is preserved we develop the notion of closure of a poset.

DEFINITION 11.1. Let $\mathbb{P}$ be a poset. $\mathbb{P}$ is **countably closed** if for every sequence of elements $\langle p_n \mid n < \omega \rangle$ of $\mathbb{P}$ such that $p_{n+1} \leq p_n$ for all n, there is $p \in \mathbb{P}$ such that $p \leq p_n$ for all n.

It is clear from the definition of $\mathbb{P}$ that it is countably closed; we just take the union of the conditions.

Lemma 11.2. *If $\mathbb{P}$ is a countably closed poset and G is $\mathbb{P}$-generic over M, then* $\omega_1^M = \omega_1^{M[G]}$.

Proof. Assume for a contradiction that there is some $p \in \mathbb{P}$ and a $\mathbb{P}$-name $\dot{f}$ which is forced by p to be a function from ω onto ω_1^M. Let $n < \omega$; we claim that $D_n = \{p \in \mathbb{P} \mid p \Vdash \dot{f}(n) = \check{\alpha} \text{ for some } \alpha < \omega_1\}$ is dense in $\mathbb{P}$. Let $p \in \mathbb{P}$ and let G be $\mathbb{P}$-generic with $p \in G$. In $M[G]$ there is an ordinal $\alpha < \omega_1^M$ such that $\dot{f}[G](n) = \alpha$. Choose $p' \in G$ forcing that $\dot{f}(n) = \check{\alpha}$. Since G is a filter we can choose $p'' \leq p', p$. Clearly $p'' \in D_n$.

By induction build a decreasing sequence of elements of $\mathbb{P}$. Let $p_0 = p$. Given p_n let $p_{n+1} \in D_n$ with $p_{n+1} \leq p_n$ and record the value α_n witnessing $p_{n+1} \in D_n$. Let $p_\omega \leq p_n$ for all n, by the countable closure of $\mathbb{P}$. Let $\alpha = \sup \alpha_n$. Let H be $\mathbb{P}$-generic over M. Then in $M[H]$, the range of $\dot{f}[H]$ is bounded by α; but this is a contradiction since it was supposed to be forced by $p \geq p_\omega \in H$ that f was onto. ⊣

A similar argument shows the following.

Lemma 11.3. *If $\mathbb{P}$ is countably closed, then whenever G is $\mathbb{P}$-generic over M,* $\mathcal{P}(\omega)^M = \mathcal{P}(\omega)^{M[G]}$.

Proof. Exercise. ⊣

There is a general phenomenon occurring in the previous proof. Suppose that $\dot{x}$ is a $\mathbb{P}$-name for an element of M. We say that a condition p **decides the value of** $\dot{x}$ if it forces $\dot{x} = \check{y}$ for some $y \in M$. The collection of conditions which decide the value of such an $\dot{x}$ is *always* dense.

Next we show the following.

Lemma 11.4. *If G is $\mathbb{P}$-generic where $\mathbb{P} = \{p \mid p : \alpha \to 2$ for some $\alpha < \omega_1\}$ ordered by extension, then* $M[G] \vDash \mathrm{CH}$.

Proof. Using the generic object G we define a list of $\omega_1^{M[G]} = \omega_1^M$-many subsets of ω. We then do a density argument to show that this list comprises all subsets of ω in $M[G]$. Work for the moment in $M[G]$. Let $g = \bigcup G$. Note that g is a function from ω_1 to 2. We define a collection of subsets of ω, $\{x_\alpha \mid \alpha < \omega_1\}$, by $n \in x_\alpha$ if and only if $g(\omega \cdot \alpha + n) = 1$.

By the previous lemma it is enough to show that for every $x \in (\mathcal{P}(\omega))^M$, there is an α such that $x = x_\alpha$. For this we will do a density argument. Work in M and let $x \subseteq \omega$. We claim $D_x = \{p \in \mathbb{P} \mid \text{there is } \alpha < \omega_1 \text{ such that for all } n,\ \chi_x(n) = p(\omega \cdot \alpha + n)\}$ is dense. (Here χ_x is the characteristic function of x.) Let $p \in \mathbb{P}$. Let $\mathrm{dom}(p) = \beta$. Let $\alpha > \beta$. It follows that for all $n < \omega$, $\omega \cdot \alpha + n \notin \mathrm{dom}(p)$. So we extend p to a condition p' in D_x where α is the witness.

It follows that in $M[G]$ the map $\alpha \mapsto x_\alpha$ is a surjection from ω_1 onto $\mathcal{P}(\omega)$. ⊣

So we have proved that $\mathrm{ZFC} + \mathrm{CH}$ is consistent.

11.2. The consistency of $\neg$CH. In this section we prove that there is a poset $\mathbb{P}$ such that whenever $M \vDash$ CH and G is $\mathbb{P}$-generic, $\omega_1^M = \omega_1^{M[G]}$ and $M[G] \vDash 2^\omega = \omega_2$. Again the poset is easy to describe. We let $\mathbb{P} = \{p \mid$ there is $x \subseteq \omega_2$ finite such that $p : x \to 2\}$ ordered by extension.

We will show that this forcing preserves all cardinals by showing that it has the *countable chain condition.* Before showing that $\mathbb{P}$ is ccc, we show that any forcing which has the ccc preserves all cardinals.

LEMMA 11.5. *Suppose that $\mathbb{P}$ is a ccc poset. Whenever G is $\mathbb{P}$-generic over M and κ is an ordinal, $M \vDash$ "κ is a cardinal" if and only if $M[G] \vDash$ "κ is a cardinal".*

PROOF. Let G be $\mathbb{P}$-generic over M and κ be an ordinal. Notice that the reverse direction is clear. So suppose that $M \vDash \kappa$ is a cardinal, but $M[G] \vDash \kappa$ is not a cardinal. Then there is a name $\dot{f}$ such that $\dot{f}[G]$ is a surjection from some $\alpha < \kappa$ onto κ. We fix a condition $p_0 \in G$ forcing this.

For every $\beta < \alpha$, the collection $D_\beta = \{p \in \mathbb{P} \mid p$ decides $\dot{f}(\beta)\}$ is dense below p_0, since $\dot{f}(\beta)$ is forced by p_0 to be an ordinal. So if we choose $A_\beta \subseteq D_\beta$ a maximal antichain, then there is a countable set of ordinals X_β such that whenever $p \in A_\beta$ there is an ordinal $\gamma \in X_\beta$ such that $p \Vdash \dot{f}(\beta) = \gamma$. But this means that $p_0 \Vdash \operatorname{ran}(\dot{f}) \subseteq \bigcup_{\beta<\alpha} X_\beta$ and the right hand union has size at most $\omega \cdot |\alpha| < \kappa$, contradicting that p_0 forces that $\dot{f}$ is onto κ. ⊣

We now recall some homework problems which will be used in showing that $\mathbb{P}$ is ccc.

Let κ be a regular cardinal.

DEFINITION 11.6. A set $C \subseteq \kappa$ is **club** if it is unbounded in κ and for all $\alpha < \kappa$ if $C \cap \alpha$ is unbounded in α, then $\alpha \in C$.

LEMMA 11.7. *The collection of club subsets of κ form a κ-complete filter.*

Recall that a filter is κ-complete if it is closed under intersections of size less than κ.

DEFINITION 11.8. A set $S \subseteq \kappa$ is **stationary** if for every club C in κ, $S \cap C \neq \varnothing$.

LEMMA 11.9. *Let S be a stationary set. If $F : S \to \kappa$ is a function such that $F(\alpha) < \alpha$ for all $\alpha \in S$, then there is a stationary $S' \subseteq S$ on which F is constant.*

LEMMA 11.10. *If S is stationary in κ, then S is unbounded in κ.*

We are now ready to prove the key lemma which will be used in the proof that $\mathbb{P}$ is ccc. We prove a weak version of this lemma which is strong enough for our application. The proof we have chosen is one that generalizes to more complicated versions of the lemma.

LEMMA 11.11 (The Δ-system lemma). *Let X be a set of size ω_1 and $\{x_\alpha \mid \alpha < \omega_1\}$ be a collection of finite subsets of X. There are an unbounded $I \subseteq \omega_1$ and a finite $r \subseteq X$ such that for all $\alpha, \beta \in I$, $x_\alpha \cap x_\beta = r$.*

The collection of sets $\{x_\alpha \mid \alpha \in I\}$ forms a Δ-system with root r.

Proof. First note that it is enough to show the lemma in the case $X = \omega_1$. Since for an arbitrary X of size ω_1 we can use a bijection with ω_1 to copy the problem. So let $\{x_\alpha \mid \alpha < \omega_1\}$ be a collection of finite subsets of ω_1.

We define a function $F : \mathrm{Lim}(\omega_1) \to \omega_1$ by $F(\alpha) = \max(x_\alpha \cap \alpha)$. Since each α is finite, we have $F(\alpha) < \alpha$ for all limit ordinals α. It follows that there are $S \subseteq \mathrm{Lim}(\omega_1)$ and $\delta < \omega_1$ such that for all $\alpha \in S$, $F(\alpha) = \delta$. Since there are only countably many finite subsets of δ, we can choose $J \subseteq S$ unbounded and a finite $r \subseteq \delta$ such that for all $\alpha \in J$, $x_\alpha \cap \delta = r$.

Finally we construct I an unbounded subset of J by recursion. Suppose that we have constructed an enumeration γ_α of I for all $\alpha < \beta$. The set $\bigcup_{\alpha<\beta} x_{\gamma_\alpha}$ is countable and hence bounded in ω_1 by some ordinal $\eta < \omega_1$. Let γ_β be the least member of J greater than η.

Now we claim that $\{x_\alpha \mid \alpha \in I\}$ forms a Δ-system with root r. Let $\alpha < \beta < \omega_1$. We will show that $x_{\gamma_\alpha} \cap x_{\gamma_\beta} = r$. By the choice of γ_β, $x_{\gamma_\alpha} \subseteq \gamma_\beta$. So $x_{\gamma_\alpha} \cap x_{\gamma_\beta} = x_{\gamma_\alpha} \cap x_{\gamma_\beta} \cap \gamma_\beta$. But $x_{\gamma_\beta} \cap \gamma_\beta = x_{\gamma_\beta} \cap \delta = r$. So we are done. ⊣

Recall the definition of $\mathbb{P}$. $\mathbb{P} = \{p \mid \text{there is a finite } x \subseteq \omega_2 \text{ such that } p : x \to 2\}$ ordered by extension.

Lemma 11.12. *$\mathbb{P}$ has the $\aleph_1$-Knaster property.*

Proof. Let $\{p_\alpha \mid \alpha < \omega_1\}$ be a sequence of conditions in $\mathbb{P}$. For each $\alpha < \omega_1$, let $x_\alpha = \mathrm{dom}(p_\alpha)$ and let $X = \bigcup_{\alpha<\omega_1} x_\alpha$. By the Δ-system lemma, there are an unbounded $I \subseteq \omega_1$ and a finite set $r \subseteq X$ such that $\{x_\alpha \mid \alpha \in I\}$ forms a Δ-system with root r.

Since there are only finitely many functions from r to 2, we can assume that for all $\alpha, \beta \in I$, $p_\alpha \restriction r = p_\beta \restriction r$. It follows that for $\alpha, \beta \in I$, $p_\alpha \cup p_\beta$ is a condition, so we are done. ⊣

Lemma 11.13. *If G is $\mathbb{P}$-generic over M, then $M[G] \vDash 2^\omega \geq \omega_2$.*

Proof. The argument is a straightforward density argument. Work in $M[G]$ and let $g = \bigcup G$. We define a collection of functions $f_\alpha : \omega \to 2$ for $\alpha < \omega_2$ by $f_\alpha(n) = 1$ if and only if $g(\omega \cdot \alpha + n) = 1$ (note that $\omega_2^M = \omega_2^{M[G]}$ since the forcing is ccc). We claim that for each pair $\alpha < \beta < \omega_2$, the set $D_{\alpha,\beta} = \{p \mid \text{there is } n \text{ such that } p(\omega \cdot \alpha + n) \neq p(\omega \cdot \beta + n)\}$ is dense. This is an argument that we have seen many times. Given a $p \in \mathbb{P}$, there is $n < \omega$ such that $\omega \cdot \alpha + n \notin \mathrm{dom}(p)$, so we are free to extend p to $p' \in D_{\alpha,\beta}$. Since $G \cap D_{\alpha,\beta} \neq \varnothing$ for all $\alpha < \beta < \omega_2$, the collection $\{f_\alpha \mid \alpha < \omega_2\}$ is a set of ω_2 many functions in ${}^\omega 2$. So $M[G] \vDash 2^\omega \geq \omega_2$. ⊣

We have shown that CH fails in $M[G]$ whenever G is $\mathbb{P}$-generic over M. We conclude by computing the value of 2^ω in the extension.

Lemma 11.14. *Let $\mathbb{P}$ be a poset and let $\dot{f}$ be a $\mathbb{P}$-name for a function from ω to 2. There is a sequence in M of functions $h_n : A_n \to 2$ for $n < \omega$, where each A_n is a maximal antichain in $\mathbb{P}$, such that whenever G is $\mathbb{P}$-generic, $\dot{f}[G](n) = h_n(p)$ where p is the unique element of $G \cap A_n$.*

PROOF. For each $n < \omega$ choose a maximal antichain A_n of elements which decide the value of $\dot{f}(n)$. Choose $h_n(p)$ to be the unique element of 2 which p decides to be the value of $\dot{f}(n)$. The conclusion is clear. ⊣

LEMMA 11.15. *If $M \models 2^\omega \leq \omega_2$ and G is $\mathbb{P}$-generic, then $M[G] \models 2^\omega \leq \omega_2$.*

PROOF. Let G be $\mathbb{P}$-generic. Every $f \in (2^\omega)^{M[G]}$ is coded by a sequence of functions as in the previous lemma. It is enough to count the number of such sequences of functions. To determine such a sequence of functions it is enough to choose an ω sequence of maximal antichains and an ω-sequence of elements of $(2^\omega)^M$. So we have at most $({\omega_2}^\omega)^\omega \cdot (2^\omega)^\omega \leq \omega_2$ objects. ⊣