FORCING EXERCISES – WEEKEND 1

Exercise -1. Get some rest.

Exercise 0. Make sure you know how to do the exercises assigned during the week.

Exercise 1. Prove that $\mathfrak{s} \leq \mathfrak{d}$.

Exercise 2. Suppose that X is uncountable and $[X]^2$ is 2-colored. Must there be an uncountable monochromatic set?

Exercise 3. Construct a non-Ramsey ultrafilter on ω.

Definition. A set $X \subseteq \mathbb{R}$ has measure zero iff for all $\epsilon > 0$ there are countably many open intervals I_n such that the diameter of I_n is less than $\epsilon/2^n$ and $X \subseteq \bigcup_{n<\omega} I_n$. (Make sure you believe this before proceeding!)

A set $X \subseteq \mathbb{R}$ is said to have **strong measure zero** iff for every sequence $\langle \epsilon_n \rangle_{n<\omega}$ of positive reals ϵ_n, there is a countable sequence $\langle I_n : n < \omega \rangle$ of open intervals such that the diameter of I_n is less than ϵ_n and $X \subseteq \bigcup_{n<\omega} I_n$.

Exercise 4.

(a) Convince yourself that the proof that $\mathbb{Q}$ has measure zero shows that $\mathbb{Q}$ also has strong measure zero.
(b) Prove that the Cantor set doesn't have strong measure zero.
(c) Prove that CH implies that there is an uncountable set of strong measure zero.
(d) $(*)$ Prove that MA implies that there is an uncountable set of strong measure zero.
(e) $(*)$ Prove that, on the other hand, MA implies that every set of size $< 2^{\aleph_0}$ does have strong measure zero.

Exercise 5 (if you know some topology)**.** Suppose that X is a topological space, F is a filter on X, and x is a point in X. We say that F **converges to** x, and write $F \to x$, if every open neighborhood of x belongs to the filter F.

(a) Let $B \subseteq X$. Prove that $x \in \overline{B}$ iff there is a filter F on X such that $B \in F$ and $F \to x$.
(b) Prove that X is compact iff every ultrafilter on X converges to at least one point.
(c) Prove that X is Hausdorff iff every ultrafilter on X converges to at most one point.
(d) Use filter-convergence and the Axiom of Choice to prove Tychonoff's theorem, that an arbitrary product of compact spaces is compact.[1]
(e) Prove (in ZF) that Tychonoff's theorem implies AC. (Tricky, but not as difficult as you'd think.)

Date: 16 July 2016.

[1] Your proof should give a proof of Tychonoff's theorem for Hausdorff spaces using the ultrafilter lemma only (not the full Axiom of Choice).